Fall 2016 - Math 3410
Name (Print): Solution KEY
Exam 2 - November 4
Time Limit: 50 Minutes

This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- If you use a "fundamental theorem" you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

| Problem | Points | Score |
| :---: | :---: | :---: |
| 1 | 20 |  |
| 2 | 12 |  |
| 3 | 12 |  |
| 4 | 12 |  |
| 5 | 24 |  |
| Total: | 80 |  |

- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

1

[^0]1. Consider the following differential equation

$$
y^{\prime \prime}+k^{2} y=0 \quad \text { where } k \text { is a constant. }
$$

(a) (3 points) Classify all points as ordinary, regular singular, or irregular singular.

Since $p(x)=0$ and $q(x)=k^{2}$ and both functions are analytic for all points, therefore all points are ordinary points.
(b) (4 points) You are going to find a power series solution to above differential equation around $x_{0}=0$. As a first step, using power series method, find the recurrence relation. By checking some of the terms, try to find a pattern.
Let $y(x)=\sum_{n=0}^{\infty} a_{n} x^{n}$. Then

$$
y^{\prime}(x)=\sum_{n=1}^{\infty} n a_{n} x^{n-1} \quad \text { and } \quad y^{\prime \prime}(x)=\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2} .
$$

Plug into the differential equation, one gets

$$
y^{\prime \prime}+k^{2} y=\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}+\sum_{n=0}^{\infty} k^{2} a_{n} x^{n}=0
$$

If we change index $n$ in the second summation we get

$$
\begin{aligned}
0 & =\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}+\sum_{n=2}^{\infty} k^{2} a_{n-2} x^{n-2} \\
& =\sum_{n=2}^{\infty}\left[n(n-1) a_{n}+k^{2} a_{n-2}\right] x^{n-2} .
\end{aligned}
$$

Since this is true for every $x$, coefficients of $x^{n-2}$ should be zero for every $n \geq 2$. Therefore,

$$
n(n-1) a_{n}+k^{2} a_{n-2}=0 \quad \text { equivalently } \quad a_{n}=\frac{-k^{2} a_{n-2}}{n(n-1)} \quad \text { for } \quad n \geq 2
$$

Now at this point, check the even terms first

$$
\begin{aligned}
a_{2} & =\frac{-k^{2} a_{0}}{2} \\
a_{4} & =\frac{-k^{2} a_{2}}{43}=\frac{k^{4} a_{0}}{432}=\frac{k^{4} a_{0}}{4!} \\
a_{6} & =\frac{-k^{2} a_{4}}{65}=\frac{-k^{6} a_{0}}{654!}=\frac{-k^{6} a_{0}}{6!} \\
\quad & \\
a_{2 n} & =\frac{(-1)^{n} k^{2 n} a_{0}}{2 n!} \text { for } n \geq 1 .
\end{aligned}
$$

Similarly, odd terms are

$$
\begin{aligned}
& a_{3}=\frac{-k^{2} a_{1}}{32} \\
& a_{5}=\frac{-k^{2} a_{3}}{54}=\frac{k^{4} a_{1}}{543!}=\frac{k^{4} a_{1}}{5!} \\
& a_{7}=\frac{-k^{2} a_{5}}{76}=\frac{-k^{6} a_{1}}{765!}=\frac{-k^{6} a_{1}}{7!} \\
& \vdots \\
& a_{2 n+1}=\frac{(-1)^{n} k^{2 n} a_{1}}{(2 n+1)!}=\frac{(-1)^{n} k^{2 n+1} a_{1}}{(2 n+1)!k} \quad \text { for } n \geq 1 .
\end{aligned}
$$

(c) (5 points) Using part (b) find the power series solution to the above differential equation. (Hint: combine even and odd terms).
Since

$$
\begin{aligned}
y(x) & =\sum_{n=0}^{\infty} a_{n} x^{n}=\text { even terms }+ \text { odd terms } \\
& =\sum_{n=0}^{\infty} a_{2 n} x^{2 n}+\sum_{n=0}^{\infty} a_{2 n+1} x^{2 n+1} .
\end{aligned}
$$

Now using part (b) we can write

$$
\begin{aligned}
y(x) & =\sum_{n=0}^{\infty} a_{2 n} x^{2 n}+\sum_{n=0}^{\infty} a_{2 n+1} x^{2 n+1} \\
& =\sum_{n=0}^{\infty} \frac{(-1)^{n} k^{2 n} a_{0}}{2 n!} x^{2 n}+\sum_{n=0}^{\infty} \frac{(-1)^{n} k^{2 n+1} a_{1}}{(2 n+1)!k} x^{2 n+1} \\
& =a_{0} \sum_{n=0}^{\infty} \frac{(-1)^{n}(k x)^{2 n}}{2 n!}+\frac{a_{1}}{k} \sum_{n=0}^{\infty} \frac{(-1)^{n}(k x)^{2 n+1}}{(2 n+1)!}
\end{aligned}
$$

which is the power series solution we are looking for.
(d) (4 points) Find the interval of convergence of the power series.

Since all points are ordinary points then there is no singularity and interval of convergence is $(-\infty, \infty)$.
(e) (4 points) Find the function representations of the power series you found in (c). From part (c) we have

$$
\begin{aligned}
y(x) & =a_{0} \sum_{n=0}^{\infty} \frac{(-1)^{n}(k x)^{2 n}}{2 n!}+\frac{a_{1}}{k} \sum_{n=0}^{\infty} \frac{(-1)^{n}(k x)^{2 n+1}}{(2 n+1)!} \\
& =a_{0} \cos (k x)+\frac{a_{1}}{k} \sin (k x) .
\end{aligned}
$$

2. (12 points) Consider the Van der Pols equation

$$
y^{\prime \prime}+\mu\left(y^{2}-1\right) y^{\prime}+y=0
$$

arose as an idealized description of a spontaneously oscillating circuit. Use the second method to find first four terms of the power series solution $y(x)=\sum_{n=0}^{\infty} a_{n} x^{n}$ around $x_{0}=0$ with $\mu=1$ with the initial conditions $y(0)=0$ and $y^{\prime}(0)=1$. $y(x)$ can also be written as its Taylor series expansion (one can it exists around $x_{0}$ );

$$
y(x)=\sum_{n=0}^{\infty} a_{n} x^{n}=\sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^{n}
$$

Therefore, we need to find $y(0), y^{\prime}(0), y^{\prime \prime}(0), y^{\prime \prime \prime}(0) . y(0)=0$ and $y^{\prime}(0)=1$ are given. To find $y^{\prime \prime}(0)$, we use the differential equation, evaluate at $x=0$ in the differential equation and use $\mu=1$ to get

$$
y^{\prime \prime}(0)+\left(y^{2}(1)-1\right) y^{\prime}(1)+y(1)=0
$$

If you solve for $y^{\prime \prime}(0)$ one gets $y^{\prime \prime}(0)=1$. In order to find $y^{\prime \prime \prime}(0)$ we differentiate the differential equation with respect to $x$ to get

$$
y^{\prime \prime \prime}+\left(y^{2}-1\right) y^{\prime \prime}+2 y y^{\prime} y^{\prime}+y^{\prime}=0
$$

Now evaluate $x=0$ to get

$$
y^{\prime \prime \prime}(0)+\left(y^{2}(0)-1\right) y^{\prime \prime}(0)+2 y(0) y^{\prime}(0) y^{\prime}(0)+y^{\prime}(0)=0
$$

Solve for $y^{\prime \prime \prime}(0)$ to get $y^{\prime \prime \prime}(0)=0$.

Now one needs to be careful while writing $a_{n}$ 's. Since $a_{n}=y^{(n)}(0) / n$ !. Therefore,

$$
\begin{array}{ccc}
a_{0}=y(0)=0 & a_{1}=y^{\prime}(0)=1 & a_{2}=y^{\prime \prime}(0) / 2=1 / 2, \\
y^{\prime \prime \prime}(0)=0 \\
a_{0}=0 & a_{1}=1 & a_{2}=1 / 2
\end{array}
$$

Using your previous work, write the first four terms of $y(x)=0+x+\frac{x^{2}}{2}+0+$.
3. (12 points) Classify the point $x_{0}=1$ as ordinary point, regular singular point, or irregular singular point for the following differential equation

$$
(x-1)^{3} y^{\prime \prime}-(x-1) y^{\prime}+4(x-1) y=0 .
$$

If we rewrite the differential equation as

$$
y^{\prime \prime}-\frac{(x-1)}{(x-1)^{3}} y^{\prime}+4 \frac{(x-1)}{(x-1)^{3}} y=0 .
$$

Hence

$$
p(x)=-\frac{(x-1)}{(x-1)^{3}} \quad \text { and } \quad q(x)=4 \frac{(x-1)}{(x-1)^{3}} .
$$

$p(x)$ and $q(x)$ are analytic at every point except the singularity point $x=1$. Hence all points are ordinary except $x=1$.

Now we check the singular point $x=1$. For this we need to check if both limits

$$
\lim _{x \rightarrow 1}(x-1) p(x) \text { and } \lim _{x \rightarrow 1}(x-1)^{2} q(x)
$$

exist and are finite. Since

$$
\lim _{x \rightarrow 1}(x-1) p(x)=\lim _{x \rightarrow 1}(x-1)\left(-\frac{(x-1)}{(x-1)^{3}}\right)=\lim _{x \rightarrow 1} \frac{1}{(x-1)}=\text { does not exist. }
$$

Therefore, $x=1$ is irregular singular point.
4. Consider the following differential equation

$$
2 x y^{\prime \prime}+y^{\prime}+x y=0 .
$$

(a) (6 points) Find the indicial equation

As we only need to find the indicial equation it is enough to plug in only the lowest term $x^{r}$ (the $n=0$ term) into the differential equation and then keep only the lowest power of $x$ that remains

$$
2 x r(r-1) x^{r-2}+r x^{r-1}+x x^{r}=2 r(r-1) x^{r-1}+r x^{r-1}+x^{r+1}=0
$$

Hence the lowest power of $x$ is $r-1$ and the coefficient is $2 r(r-1)+r=0$ is the indicial equation. That is,

$$
r(2(r-1)+1)=r(2 r-1)=0 .
$$

(b) (6 points) Write the general form of the solution(s).

Since $0=r(2 r-1)$, the roots of the indicial equation are $r_{1}=0$ and $r_{2}=1 / 2$. Since $r_{1}-r_{2}=-1 / 2$ is not integer then the method of Frobenious tells us that there are two linearly independent solutions

$$
y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}=\sum_{n=0}^{\infty} a_{n} x^{n}
$$

is the first linearly independent solution and the second solution is

$$
y_{2}(x)=\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}=x^{\frac{1}{2}} \sum_{n=0}^{\infty} b_{n} x^{n}
$$

5. Consider the function $f(x)=|x|$ on $[-\pi, \pi]$ and $f(x+2 \pi)=f(x)$.
(a) (2 points) What is the period $2 L$ of $f(x) ?$

Solution: Since $f(x+2 \pi)=f(x)$ is given then $f(x)$ has period $2 \pi=2 L$, i.e., $2 L=2 \pi$.
Hence $L=\pi$ for later use.
(b) (2 points) Is $f(x)$ an odd or even function? Show your work.

Solution: Since $f(x)=f(-x), f$ is an even function.
(c) (5 points) Find the sine terms of the Fourier series of $f(x)$.

Solution: Since $f(x)$ is an even function then all sine terms are zero;

$$
b_{n}=0 \quad \text { for all } n=1, \ldots,
$$

(d) (5 points) Find the cosine terms of the Fourier series of $f(x)$.

Solution: The cosine terms are

$$
a_{0}=\frac{2}{\pi} \int_{0}^{\pi} x d x=\frac{\pi}{2}
$$

Now

$$
\begin{aligned}
a_{n} & =\frac{2}{\pi} \int_{0}^{\pi} x \cos \left(\frac{n \pi x}{L}\right) d x=\frac{2}{\pi} \int_{0}^{\pi} x \cos \left(\frac{n \pi x}{\pi}\right) d x=\frac{2}{\pi} \int_{0}^{\pi} x \cos (n x) d x \\
& =\left.\frac{2}{\pi n}(x \sin (n x))\right|_{0} ^{\pi}-\frac{2}{\pi n} \int_{0}^{\pi} \sin (n x) d x \\
& =0+\frac{2}{\pi n^{2}}(\cos (\pi n)-1) \\
& =\frac{2}{\pi n^{2}}\left((-1)^{n}-1\right)
\end{aligned}
$$

Note that $a_{n}=0$ when $n$ is even and $a_{n}=-4 / \pi n^{2}$ when $n$ is odd. Hence $a_{2 k-1}=$ $-4 / \pi(2 k-1)^{2}$ for $k=1, \ldots$,
(e) (4 points) Write the Fourier series of the function $f(x)$.

Solution: From parts (c) and (d) we have

$$
\begin{aligned}
f(x) & =\frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos (n \pi x / \pi)+\sum_{n=1}^{\infty} b_{n} \sin (n \pi x / \pi) \\
& =\frac{\pi}{2}-\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2 k-1)^{2}} \cos ((2 k-1) x) .
\end{aligned}
$$

(f) (3 points) Using Fourier series convergence theorem, check the points where $F(x)$ and $f(x)$ agree and do not agree.
Solution: Since $f(x)=|x|$ is continuous on $(-\pi, \pi)$ then by Fourier series convergence theorem $f(x)=F(x)$. Moreover, one can also check $F(x)=f(x)$ when $x=\pi,-\pi$. Hence $F(x)=f(x)=|x|$. They agree everywhere.
(g) (3 points) Use part (e) and verify that

$$
\sum_{n=0}^{\infty} \frac{1}{(2 n+1)^{2}}=\frac{\pi^{2}}{8} .
$$

Solution: Note that when $x=0$, the function $f$ and $f^{\prime}$ are both continuous therefore, the Fourier series converges to $f(0)$ at $x=0$. Since $f(0)=0$ using this and evaluating $x=0$ in the Fourier series of $f$ one gets

$$
0=f(0)=\frac{\pi}{2}-\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2 k-1)^{2}} \cos (0) .
$$

This implies

$$
\sum_{n=0}^{\infty} \frac{1}{(2 n+1)^{2}}=\frac{\pi^{2}}{8}
$$

SCRATCH PAPER


[^0]:    ${ }^{1}$ Exam template credit: http://www-math.mit.edu/~psh

