Fall 2016 - Math 3410
Final Exam - December 14
Time Limit: 120 Minutes

Name (Print): ____________________

This exam contains 14 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	12	
2	36	
3	36	
4	24	
5	12	
6	0	
Total:	120	

Do not write in the table to the right.

[1]

[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. Consider the following Laplace's equation in rectangle

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, \quad 0 < x < 3, \quad 0 < y < 3, \\ u_x(0, y) = -3 \quad \text{and} \quad u_x(3, y) = 0, \\ u_y(x, 0) = 3 \quad \text{and} \quad u_y(x, 3) = 0 \end{cases}$$

and consider the following function

$$u(x, y) = \frac{1}{2}(x^2 - y^2) - 3x + 3y + 3410.$$

(a) (6 points) Verify that $u(x, y)$ satisfies the Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$.
Solution: We need to compute u_{xx} and u_{yy};

$$u_x = x - 3 \quad \text{and} \quad u_{xx} = 1.$$

Similarly,

$$u_y = -y + 3 \quad \text{and} \quad u_{yy} = -1.$$

Hence

$$u_{xx} + u_{yy} = 1 - 1 = 0.$$

Therefore, we conclude that given function $u(x, y)$ satisfies the Laplace's equation.

(b) (6 points) Verify also that $u(x, y)$ satisfies the given boundary conditions as well (check each of them separately) and conclude that $u(x, y)$ solves the above Laplace's equation with the given boundary conditions.
Solution: We need to check $u(x, y)$ satisfies all of the four boundary conditions;
First boundary condition $u_x(0, y) = -3$;

$$u_x(x, y) = x - 3 \quad \text{therefore} \quad u_x(0, y) = 0 - 3 = -3.$$

Second boundary condition $u_x(3, y) = 0$;

$$u_x(x, y) = x - 3 \quad \text{therefore} \quad u_x(3, y) = 3 - 3 = 0.$$

Third boundary condition $u_y(x, 0) = 3$;

$$u_y(x, y) = -y + 3 \quad \text{therefore} \quad u_y(x, 0) = -0 + 3 = 3.$$

Fourth boundary condition $u_y(x, 3) = 0$;

$$u_y(x, y) = -y + 3 \quad \text{therefore} \quad u_y(x, 3) = -3 + 3 = 0.$$

Hence $u(x, y)$ satisfies all the boundary conditions. We then conclude that $u(x, y)$ is a solution to the Laplace's equation with the given boundary values.

2. Consider the following Heat conduction problem

$$\begin{cases} 4u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = 0 \quad \text{and} \quad u_x(\pi,t) = 0 \\ u(x,0) = \cos(3x) + \cos(4x) + \cos(10x). \end{cases}$$

(a) (6 points) By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

Solution: Rewrite the PDE as $4u_{xx} - u_t = 0$. Let $u(x,t) = X(x)T(t)$. Then

$$u_{xx} = X''T \quad \text{and} \quad u_t = XT'.$$

Substitute this in to the differential equation to get

$$4u_{xx} - u_t = 4X''T - XT' = 0 \quad \text{equivalently} \quad \frac{X''}{X} = \frac{T'}{4T} = -\lambda.$$

Hence

$$\begin{aligned} \frac{X''}{X} = -\lambda \quad &\to \quad X'' + \lambda X = 0, \\ \frac{T'}{4T} = -\lambda \quad &\to \quad T' + 4\lambda T = 0. \end{aligned}$$

(b) (4 points) Rewrite the boundary values in terms of X and T and choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.

Solution: We have at $x = 0$ as $u_x(x,t) = X'(x)T(t)$ then

$$u_x(0,t) = X'(0)T(t) = 0; \quad \text{one has either} \quad X'(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = \pi$

$$u_x(\pi,t) = X'(\pi)T(t) = 0; \quad \text{one has either} \quad X'(\pi) = 0 \quad \text{or} \quad T(t) = 0.$$

We know that the choice of $T(t) = 0$ gives only the trivial solution as $u(x,t) = X(x)T(t) = 0$.

Therefore, we choose our boundary conditions as $X'(0) = 0$ and $X'(\pi) = 0$ in order to obtain the non-trivial solution. Now if we rewrite the ordinary differential equation corresponding to X we get

$$X'' + \lambda X = 0, \quad X'(0) = 0 \quad \text{and} \quad X'(\pi) = 0.$$

(c) (8 points) Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigen-functions X_n

Solution: For $\lambda = 0$ we get $X(x) = b$ is a solution. Hence we will keep this as eigenfunction.

For $\lambda < 0$ we trivial solution. (solution will be exponential and is not period).

For $\lambda > 0$ We know that the non-trivial general solutions is

$$X(x) = A\cos(kx) + B\sin(kx)$$

where we choose $\lambda = k^2$, $k > 0$. Using this general solution and the first boundary condition and $X'(x) = -Ak\sin(kx) + Bk\cos(kx)$

$$X'(0) = -Ak\sin(0) + Bk\cos(0) = Bk = 0 \quad \text{therefore we have} \quad B = 0.$$

Using the second boundary condition, we get (as $B = 0$, $X'(x) = -Ak\sin(kx)$)

$$X'(\pi) = -Ak\sin(k\pi) = 0.$$

This holds when $k\pi = n\pi$ for $n = 1, 2, \ldots$. Hence we get $k = n$, or equivalently, we get the eigenvalues

$$\lambda_n = k^2 = n^2 \quad n = 1, 2, \ldots.$$

The corresponding eigenfunction (corresponding to λ_n) is

$$X_0(x) = b, \quad \text{and} \quad X_n(x) = \cos(nx) \quad \text{for } n = 1, 2, \ldots.$$

(d) (8 points) For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

Solution: Since the ordinary differential equation corresponding to T is

$$T' + 4\lambda T = 0.$$

Plug in $\lambda = n^2$ we get (for each n we have a different solution T_n)

$$T_n' + 4n^2 T_n = 0.$$

We know that this is a first order linear ordinary differential equation and its solution is

$$T_n(t) = C_n e^{-4n^2 t}.$$

for some C_n.

For $\lambda = 0$, we get $T' = 0$, then $T(t) = C_0$. Hence this is also solution corresponding to $\lambda = 0$ we will also keep this.

(e) (5 points) Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

Solution: We know that the solution for each n is

$$u_n(x) = X_n(x)T_n(t) = \cos(nx)C_n e^{-4n^2 t} \text{ for } n = 1, 2, \ldots, \quad \text{and } u_0(x,t) = b_0 C_0.$$

The general solution is

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = b_0 C_0 + \sum_{n=1}^{\infty} C_n \cos(nx) e^{-4n^2 t}.$$

(f) (5 points) Using the given initial value and the general solution you found in (f), find the particular solution.

Solution: Now the initial value is given as $u(x,0) = \cos(3x) + \cos(4x) + \cos(10x)$. Hence, plug in $t = 0$ in the solution we have in (f) gives us

$$u(x,0) = b_0 C_0 + \sum_{n=1}^{\infty} C_n \cos(nx) e^{0} = \sum_{n=1}^{\infty} C_n \cos(nx) = \cos(3x) + \cos(4x) + \cos(10x).$$

From this we get that, all $C_n = 0$ except $C_3 = 1$, $C_4 = 1$, and $C_{10} = 1$, here there is no constant term, therefore, $b_0 C_0 = 0$. Hence in the solution, the only terms we have, for $n = 2, n = 4, n = 10$ with $C_3 = 1$, $C_4 = 1$, and $C_{10} = 1$;

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n \cos(nx) e^{-4n^2 t} = \cos(3x) e^{-4\,3^2 t} + \cos(4x) e^{-4\,4^2 t} + \cos(10x) e^{-4\,10^2 t} \\ &= \cos(3x) e^{-36t} + \cos(4x) e^{-64t} + \cos(10x) e^{-400t} \end{aligned}$$

is the particular solution.

3. Consider the following wave equation which describes the displacement $u(x,t)$ of a piece of flexible string with the initial boundary value problem

$$\begin{cases} 9u_{xx} = u_{tt}, \quad 0 < x < 3, \quad t > 0 \\ u(0,t) = 0 \quad \text{and} \quad u(3,t) = 0 \\ u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 3\pi\sin(\pi x). \end{cases}$$

(a) (6 points) By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T.

Solution: Rewrite the differential equation as $9u_{xx} - u_{tt} = 0$. Let $u(x,t) = X(x)T(t)$. Then we get

$$u_{xx} = X''T \quad \text{and} \quad u_{tt} = XT''.$$

Substitute this into the partial differential equation $9u_{xx} - u_{tt} = 0$ to get

$$9u_{xx} - u_{tt} = 9X''T - XT'' = 0.$$

Dividing by $4XT$ we get

$$\frac{X''}{X} = \frac{T''}{9T}.$$

Notice that the left-hand side is a function of x only and the right-hand side is function of t only and as they are same, this is possible only if they are the same constant;

$$\frac{X''}{X} = \frac{T''}{9T} = -\lambda.$$

From this we get

$$\begin{aligned} \frac{X''}{X} &= -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \\ \frac{T''}{9T} &= -\lambda \quad \rightarrow \quad T'' + 9\lambda T = 0. \end{aligned}$$

(b) (4 points) Rewrite the boundary values in terms of X and T and choose the boundary values which will not give a non-trivial solution and then rewrite the ordinary differential equation with boundary values corresponding to X.

Solution: Since $u(x,t) = X(x)T(t)$ we have when $x = 0$

$$u(0,t) = X(0)T(t) = 0 \quad \text{we should have either} \quad X(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = 3$, we have

$$u(3,t) = X(3)T(t) = 0 \quad \text{we should have either} \quad X(3) = 0 \quad \text{or} \quad T(t) = 0.$$

We know that $T(t) = 0$ will give only the trivial solution. Therefore, we should choose

$$X(0) = 0 \quad \text{and} \quad X(3) = 0.$$

Then X satisfies the following two-point boundary condition

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(3) = 0.$$

(c) (8 points) Solve the two-point boundary value problem corresponding to X you found in (b). Find all eigenvalues λ_n and eigenfunctions X_n

Solution: Now we know that only non-trivial solution comes from when $\lambda = k^2$ for some $k > 0$ (again when $\lambda = 0$ and $\lambda < 0$ will give only the trivial solution, $X(x) = 0$). In this case the solution is

$$X(x) = A\cos(kx) + B\sin(kx).$$

We now find A and B using the boundary values we have in (c), at $x = 0$

$$X(0) = A\cos(0) + B\sin(0) = 0 \quad \text{implies} \quad A = 0.$$

At $x = 3$, (now $A = 0$, we only have $X(x) = B\sin(kx)$)

$$X(0) = B\sin(3k) = 0.$$

This holds if $\sin(3k) = 0$ which gives $3k = \pi n$. Hence $k = \pi n/3$.

$$k = \frac{\pi n}{3} \quad \text{hence} \quad \lambda = k^2 = \frac{\pi^2 n^2}{3^2}.$$

Therefore, the eigenvalues are

$$\lambda_n = \frac{\pi^2 n^2}{3^2}.$$

The corresponding eigenfunctions are

$$X_n(x) = \sin(kx) = \sin(\frac{\pi n x}{3}) \quad n = 1, 2, \ldots.$$

(d) (8 points) For each eaigenvalue λ_n you found in (c), solve the initial value problem corresponding to T_n.

Solution: Since $\lambda_n = \frac{\pi^2 n^2}{3^2}$ and the ordinary differential equation corresponding to T is

$$T'' + 9\lambda T = 0.$$

Substitute $\lambda =$ we get (we have a different solution for each n);

$$T_n'' + 9\lambda T_n = T_n'' + 9\frac{\pi^2 n^2}{3^2} T_n = T_n'' + \pi^2 n^2 T_n$$

This is a second order linear differential equation and which has characteristic equation

$$r^2 + \pi^2 n^2 = 0.$$

From this we get that the characteristic equation has a imaginary complex conjugate roots

$$r = \pm \pi n \mathbf{i}.$$

Thus the solutions are

$$T_n(t) = A_n\cos(\pi n t) + B_n\sin(\pi n t) \quad n = 1, 2, \ldots, .$$

(e) (5 points) Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.
Solution: Combining (c) and (e) we have

$$u_n(x,t) = X(x)T(t) = \sin(\frac{\pi n x}{3})[A_n \cos(\pi n t) + B_n \sin(\pi n t)].$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{3})[A_n \cos(\pi n t) + B_n \sin(\pi n t)].$$

(f) (5 points) Using the given initial values, find the particular solution.
Solution: Now we have two initial values $u(x,0) = 0$ and $u_t(x,0) = 3\pi \sin(3x)$
At $t = 0$, i.e, $u(x,0) = 0$

$$u(x,0) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{3})[A_n \cos(0) + B_n \sin(0)] = 0$$

gives us $A_n = 0$ for every $n = 1, 2, \ldots$. Hence we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{3}) B_n \sin(\pi n t).$$

we now need to find $u_t(x,t)$;

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{3}) B_n \pi n \cos(\pi n t).$$

Then given initial value $u_t(x,0) = 3\pi \sin(3x)$ gives us

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{3}) B_n \pi n \cos(0) = 3\pi \sin(3x).$$

Therefore, let $b_n = B_n \pi n$ then above identity becomes

$$\sum_{n=1}^{\infty} b_n \sin(\frac{\pi n x}{3}) = \sin(\pi x).$$

This tells us all $b_n = 0$ except $b_3 = 1$ as

$$\sum_{n=1}^{\infty} b_n \sin(\frac{\pi n x}{3}) = b_1 \sin(\frac{\pi x}{3}) + b_2 \sin(\frac{\pi 2x}{3}) + b_3 \sin(\frac{\pi 3x}{3}) + \ldots$$

and we know that this summation is $\sin(3x)$ hence all b_n should be zero except b_3 which will be 3π. Hence if we solve for B_5 here, (we plug in $n = 3$) $3\pi = b_3 = B_3 3\pi$. Hence

$B_3 = 1$

$$u(x,t) = B_3 \sin(\frac{\pi 3x}{3}) \sin(\pi 3t) = \sin(\pi x) \sin(3\pi t)$$

is the particular solution.

4. For the following differential equation

$$xy'' + y' - y = 0$$

(a) (4 points) Find and classify all points as ordinary, regular singular, or irregular singular points.

Solution: Rewrite the differential equation as

$$y'' + \frac{y'}{x} - \frac{y}{x} = 0$$

Therefore, $p(x) = 1/x$ and $q(x) = -1/x$ are both singular at $x = 0$. We conclude that all points except $x = 0$ is ordinary points. For $x = 0$ we need to check the following limits

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{x} = 0.$$

Since both limits exist and are finite, therefore $x = 0$ is regular singular points.

(b) (5 points) For each of the regular point(s), find the corresponding indicial equation and find the double root r_1 of the indicial equation (Yes there is one double root).

Solution: Since $x = 0$ is the only regular singular point, we find the corresponding indicial equation. Let

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

be a solution for some r and a_n. Find y' and y'' in terms of power series.

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Plug in to the differential equation to get

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

After some algebra one gets

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

In order to write the three summation under one sum we need to change the power of x from $n+r$ to $n+r-1$ so that the powers of x in each summation match (one can change the power of x from $n+r-1$ to $n+r$ in the first two summation, idea is the same). Therefore,

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0.$$

We can rewrite the power series as (just split the $n = 0$ terms and leave the remaining)

$$r(r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+ra_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0.$$

This gives us

$$(r(r-1)a_0+ra_0)x^{r-1}+\sum_{n=1}^{\infty}[(n+r)(n+r-1)a_n+(n+r)a_n-a_{n-1}]x^{n+r-1}=0$$

Since this is true for every x we get (assuming again $a_0 \neq 0$)

$$r(r-1)+r=0 \quad \text{and} \quad (n+r)(n+r-1)a_n+(n+r)a_n-a_{n-1}=0 \quad \text{for} \quad n \geq 1. \quad (1)$$

Therefore the indicial equation is $r^2 = 0$. Hence we have a double root $r_1 = 0$.

(c) (5 points) Find the corresponding recurrence relation for the root r_1.
Solution: To find the corresponding recurrence relation corresponding to $r_1 = 0$, simply plug in $r = 0$ in (1) to get

$$a_n = \frac{a_{n-1}}{n^2} \quad \text{for} \quad n \geq 1.$$

(d) (5 points) Find the corresponding power series solution y_1.
Solution: As we assume that $a_0 \neq 0$ we get

$$\begin{aligned}
a_1 &= a_0,\\
a_2 &= \frac{a_1}{2^2} = \frac{a_0}{2^2} = \frac{a_0}{(2!)^2}\\
a_3 &= \frac{a_2}{3^2} = \frac{a_0}{3^2 2^2} = \frac{a_0}{(3!)^2}\\
&\dots\\
a_n &= \frac{a_0}{(n!)^2}.
\end{aligned}$$

Therefore, we obtain the first solution corresponding to $r_1 = 0$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n.$$

(e) (5 points) Use the method of Frobenious and write down the general form of the second solution y_2.
Solution: Since the indicial equation has double root, the method of Frobenious tells us that the second linearly independent solution is of the form

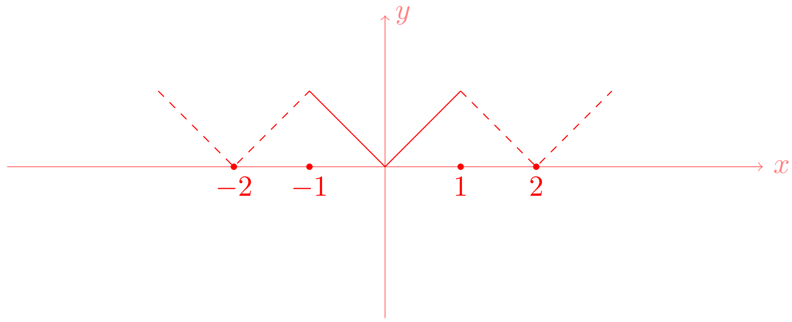
$$y_2(x) = \sum_{n=1}^{\infty} b_n x^{n+r} + \log(x) y_1(x) = \sum_{n=0}^{\infty} b_n x^n + \log(x) y_1(x)$$

5. Let $f(x)$ be given as

$$f(x) = x, \quad 0 < x < 1.$$

(a) (6 points) Extend $f(x)$ into an even periodic function with period of 2.

Solution: As we want to extend f into odd periodic function with period of 2 we define



Odd extension $F_{\text{even}}(x)$ of f is, $F_{\text{even}}(x+2) = F_{\text{even}}(x)$ and

$$F_{\text{even}}(x) = \begin{cases} f(x) & 0 \leq x < 1, \\ f(-x) & -1 < x \leq 0. \end{cases} = \begin{cases} x & 0 \leq x < 1, \\ (-x) & -1 < x \leq 0 \end{cases}$$

(b) (6 points) Find Fourier series $F(x)$.

Solution: As we have the extension F_{even}, which is periodic with period of $2L = 2$, (hence $L = 1$) we will find its Fourier series.

Since this is an even extension, sine terms will be zero; $b_n = 0$. Hence it will be a cosine series; we only need to find a_0 and a_n.

$$a_0 = \frac{1}{L}\int_{-1}^{1} f(x)dx = 2\int_0^1 f(x)dx = 2\int_0^1 xdx = 1.$$

Now for a_n we get

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx \\ &= 2\int_0^1 x\cos(n\pi x)dx \\ &= \frac{2}{n\pi}[x\sin(n\pi x)]_{x=0}^{x=1} - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)dx \\ &= 0 + \frac{2}{n^2\pi^2}[\cos(n\pi x)]_{x=0}^{x=1} \\ &= \frac{2}{n^2\pi^2}[(-1)^n - 1] \end{aligned}$$

Hence the Fourier series of F_{even} is

$$\begin{aligned} F(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{2}) \\ &= 1 + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(n\pi x) + 0. \end{aligned}$$

6. (10 points (bonus)) Solve the first order equation

$$4u_x + u_y = 0$$

with the auxiliary condition

$$u(0, y) = \frac{1}{1 + y^2}.$$

Solution: Notice that

$$\langle (4, 1), \nabla u(x, y) \rangle = 4u_x + u_y = 0$$

Hence $u(x, y)$ is constant in the direction of $(4, 1)$. The lines parallel to $(4, 1)$ have equations $-x + 4y = 0$. here $-x + 4y = 0$ is called the characteristic lines. As $u(x, y)$ is constant on these lines therefore $u(x, y)$ depends only $-x + 4y$. Hence

$$u(x, y) = f(-x + 4y).$$

Using the the auxiliary condition $u(0, y) = \frac{1}{1+y^2}$ we get

$$u(0, y) = f(4y) = \frac{1}{1 + y^2}.$$

Since $f(4y) = \frac{1}{1+y^2}$. we can find $f(y) = \frac{1}{1+(y/4)^2}$. Since

$$u(x, y) = f(-x + 4y) = \frac{1}{1 + \frac{1}{16}(-x + 4y)^2}.$$

is the solution. (you can check you answer by finding u_x and u_y and verify that $4u_x + u_y = 0$.)