Fall 2016 - Math 3410
Practice Exam 2 - November 4
Time Limit: 50 Minutes

Name (Print): Solution KEY

This exam contains 11 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	12	
3	12	
4	12	
5	24	
Total:	80	

[1]

[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. Consider the following differential equation

$$y'' + 4y = 0.$$

(a) (3 points) Verify that $x_0 = 0$ is an ordinary point.

Solution: Since both $p(x) = 0$ and $q(x) = 4$ are polynomial and therefore analytic at every points. Hence all points are ordinary points.

(b) (4 points) Using power series method find the recurrence equation that the coefficients satisfy. Solution: Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find y'' to get

$$\begin{aligned} 0 = y'' + 4y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 4\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 4a_n]x^n. \end{aligned}$$

Therefore, the corresponding recurrence relation is $(n+2)(n+1)a_{n+2} + 4a_n = 0$ for every $n \geq 0$. Equivalently,

$$a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}.$$

(c) (5 points) Using part (b) find the power series solution to the above differential equation. (Hint: combine even and odd terms).

Solution:
Now, consider the even terms first;

$$\begin{aligned} a_2 &= \frac{-2^2 a_0}{2\,1}, \\ a_4 &= \frac{-2^2 a_2}{4\,3} = \frac{2^4 a_0}{4\,3\,2} \\ a_6 &= \frac{-2^2 a_4}{6\,5} = \frac{-2^6 a_0}{6\,5\,4\,3\,2} \\ &\dots \\ a_{2n} &= (-1)^n \frac{a_0 4^n}{(2n)!} \text{ for } n = 1, 2, \dots \end{aligned}$$

Similarly, odd terms are

$$\begin{aligned} a_3 &= \frac{-2^2 a_1}{3\,2}, \\ a_5 &= \frac{-2^2 a_1}{5\,4} = \frac{2^4 a_1}{5\,4\,3\,2} \\ a_7 &= \frac{-2^2 a_5}{7\,6} = \frac{-2^6 a_1}{7\,6\,5\,4\,3\,2} \\ &\dots \\ a_{2n+1} &= (-1)^n \frac{2^{2n+1} a_1}{2(2n+1)!} \text{ for } n = 1, 2, \dots \end{aligned}$$

Now the solution $y(x)$ can be written as even terms plus the odd terms

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}.$$

Hence

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} a_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} (-1)^k \frac{a_1 2^{2k+1}}{2(2k+1)!} x^{2k+1}. \end{aligned}$$

(d) (4 points) Find the radius of convergence of the power series.

Since both $p(x)$ and $q(x)$ are analytic for every x, i.,e, there is no singular point. Therefore, the power series converges for every x and the radius of convergence is ∞.

(e) (4 points) Find the function representations of the power series you found in (c).

Remember that $\sin(2x)$ and $\cos(2x)$ have power series representation

$$\sin(2x) = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k+1}}{(2k+1)!}x^{2k+1} \quad \text{and} \quad \cos(2x) = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}}{(2k)!}x^{2k}.$$

Now using these we get

$$y(x) = a_0 \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}}{(2k)!}x^{2k} + \frac{a_1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{2^{2k+1}}{(2k+1)!}x^{2k+1} = a_0\cos(2x) + \frac{a_1}{2}\sin(2x).$$

2. (12 points) Consider the *Rayleigh*'s equation

$$my'' + ky = ay' - b(y')^3$$

which models the oscillation of a clarinet reed. Using the second method find the first four terms of the power series solution $y(x)$ around $x_0 = 0$ with $m = k = a = 1$ and $b = 1/3$ with the initial conditions $y(0) = 0$ and $y'(0) = 1$. Write the solution $y(x)$.

It is given that $m = k = a = 1$ and $b = 1/3$, plugin these numbers into the differential equation to get

$$y'' + y = y' - \frac{1}{3}(y')^3 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

Since $x_0 = 0$ is ordinary point, $y(x)$ has Taylor series expansion around $x_0 = 0$;

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n, \quad \text{i.e.,} \quad a_n = \frac{y^{(n)}(0)}{n!}.$$

Therefore, using the second method, we find $y(0), y'(0), y''(0), y'''(0)$. Luckily, $y(0) = 0$ and $y'(0) = 1$ are given. So we need to find $y''(0)$. Use the differential equation, plugin $x = 0$ to get

$$y''(0) + y(0) = y'(0) - \frac{1}{3}(y'(0))^3 \quad \text{equivalently} \quad y''(0) = 1 - \frac{1}{3} = -\frac{2}{3}.$$

Now we need to find $y'''(0)$. To do this end, differentiate the differential equation with respect to x and evaluate at $x = 0$ to get

$$y'''(0) + y'(0) = y''(0) - (y'(0))^2 y''(0) \quad \text{equaivalently} \quad y'''(0) = -1.$$

Hence

$a_0 = 0$ $\qquad$ $a_1 = 1$ $\qquad$ $a_2 = \frac{-2}{3}\frac{1}{2!}$ $\qquad$ $a_3 = \frac{-1}{3!}$.

$$y(x) = 0 + x - \frac{1}{3}x^2 - \frac{1}{6}x^3 + \dots$$

3. (12 points) Classify the point $x_0 = 0$ as ordinary point, regular singular point, or irregular singular point for the following differential equation

$$2x^2(1+x)y'' - x(1-3x)y' + y = 0.$$

Solution: If we rewrite the differential equation as

$$y'' + \frac{-x(1-3x)}{2x^2(1+x)}y' + \frac{1}{2x^2(1+x)}y = y'' + p(x)y' + q(x)y = 0$$

then

$$p(x) = \frac{-x(1-3x)}{2x^2(1+x)} \quad \text{and} \quad q(x) = \frac{1}{2x^2(1+x)}.$$

It is obvious that $x_0 = 0$ is not ordinary point as $p(x)$ and $q(x)$ are singular at $x_0 = 0$ (i.e., derivatives of some order of either function does not exists at $x_0 = 0$). One needs to check first;

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{-x^2(1-3x)}{2x^2(1+x)} = -\frac{1}{2}.$$

Similarly,

$$\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} \frac{x^2}{2x^2(1+x)} = \frac{1}{2}.$$

Since both limits exist and are finite then $x_0 = 0$ is a regular singular point.

4. Consider the following differential equation

$$x^2y'' + x\left(x - \frac{1}{2}\right)y' + \frac{1}{2}y = 0.$$

(a) (6 points) Find the indicial equation associated to regular singular point $x_0 = 0$.

Solution: Since we only want to find the indicial equation it is enough to plug in $y(x) = x^r$ and then combine the lowest powers of x whose coefficient will be the indicial equation. Therefore, Since the indicial equation

$$x^2r(r-1)x^{r-2} + x(x-1/2)rx^{r-1} + 1/2x^r = r(r-1)x^r + rx^{r+1} - 1/2rx^r + 1/2x^r.$$

Now, the lowest power of x is r and the coefficient is $r(r-1) - r/2 + 1/2$. Hence, indicial equation is

$$r(r-1) - r/2 + 1/2 = 0.$$

(b) (6 points) Write the general form of the solution(s).

Solution: Since

$$r(r-1) - r/2 + 1/2 = 0.$$

has roots $r_1 = 1$ and $r_2 = \frac{1}{2}$ and $r_1 - r_2 = 1 - 1/2 = 1/2$ which is not integer and not zero by the method of Frobenious we know that there are two linearly independent solutions to the above differential equation. Since $r_1 = 1$ then the first linearly independent solution has the form

$$y(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = x^1 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$

for some a_n.

The second linearly independent solution has the form

$$y(x) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n = x^{1/2} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

for some b_n.

5. Consider the following piecewise function

$$f(x) = \begin{cases} -2 & -1 \le x < 0, \\ 2 & 0 \le x < 1. \end{cases} \quad \text{and} \quad f(x+2) = f(x).$$

(a) (3 points) What is the period $2L$ of $f(x)$?
Solution: Since $f(x+2) = f(x)$ is given then $f(x)$ has period $2 = 2L$, i.e., $2L = 2$. Hence $L = 1$ for later use.

(b) (3 points) Is $f(x)$ an odd or even function? Show your work.
Solution: Since $f(x) = -f(-x)$, f is an odd function.

(c) (5 points) Find the sine terms of the Fourier series of $f(x)$. The sine terms are b_n (remember $L = 1$ from part (a)) where

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-1}^{1} f(x)\sin(\pi n x/L)dx = \int_{-1}^{1} f(x)\sin(\pi n x)dx \\ &= \int_{-1}^{0} (-2)\sin(\pi n x)dx + \int_{0}^{1} 2\sin(\pi n x)dx \\ &= -2\frac{1}{\pi n}(-\cos(\pi n x))|_{x=-1}^{x=0} + 2\frac{1}{\pi n}(-\cos(\pi n x))|_{x=0}^{x=1} \\ &= \frac{2}{\pi n} - \frac{2}{\pi n}\cos(\pi n) - \frac{2}{\pi n}\cos(\pi n) \\ &= \frac{4}{\pi n}(1 - \cos(\pi n)). \end{aligned}$$

Notice that $b_n = 0$ when n is even and $b_n = 8/(\pi n)$ when n is odd. So if we rewrite $b_{2k-1} = 8/(\pi(2k-1))$ for $k = 1, \ldots, .$

(d) (5 points) Find the cosine terms of the Fourier series of $f(x)$.
Solution: Since $f(x)$ is an odd function then all cosine terms are zero;

$$a_1 = 0 \quad \text{and} \quad a_n = 0 \quad \text{for all} \quad n = 1, \ldots, .$$

(e) (4 points) Write the Fourier series of the function $f(x)$.

Solution: From parts (c) and (d) we have

$$\begin{aligned} F(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi n x) + \sum_{n=1}^{\infty} b_n \sin(\pi n x) \\ &= 0 + 0 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(\pi(2k-1)x) \\ &= \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(\pi(2k-1)x). \end{aligned}$$

I do not expect this part; Notice that $f(x)$ is piecewise function, and $f(x)$ is continuous on $(-1,0)$ and $(0,1)$. By Fourier convergence theorem, $F(x) = f(x)$ for $x \in (-1,0)$ and $x \in (0,1)$.

$$F(0) = \frac{1}{2}[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)] = \frac{1}{2}[-2+2] = 0.$$

Notice that $f(0) = 2$ which does not agree with its Fourier series $F(x)$ at $x = 0$.

Likewise, at $x = 1$ and $x = -1$

$$\begin{aligned} F(-1) &= \frac{1}{2}[\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)] = \frac{1}{2}[2-2] = 0 \\ F(1) &= \frac{1}{2}[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)] = \frac{1}{2}[2-2] = 0. \end{aligned}$$

Notice also that $f(1) = -2$ which does not agree with its Fourier series $F(x)$ at $x = 1$, as $F(1) = 0$. Similarly, $f(-1) = -2$ which does not agree with its Fourier series $F(x)$ at $x = -1$, as $F(-1) = 0$.

(f) (4 points) Use part (e) and verify that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$$

Solution: At $x = 1/2$, $f(x)$ is continuous, therefore, the Fourier series $F(x)$ agrees with $f(x)$ at $x = 1/2$. As $f(1/2) = 2$ and using this and evaluating the Fourier series of f at $x = 1/2$ one gets

$$2 = f(1/2) = F(1/2) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(\pi(2k-1)) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}.$$

From this one gets

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$$

(I do not expect this part) Just a note to say here, we normally know that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

converges as it is an alternating power series. But this question provides not only that power series converges but also gives precise number.

SCRATCH PAPER