Fall 2016 - Math 3410
Name (Print):
Practice Final Exam - December 14
Time Limit: 120 Minutes

This exam contains 14 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- If you use a "fundamental theorem" you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages;

| Problem | Points | Score |
| :---: | :---: | :---: |
| 1 | 12 |  |
| 2 | 36 |  |
| 3 | 36 |  |
| 4 | 24 |  |
| 5 | 12 |  |
| 6 | 0 |  |
| Total: | 120 |  | clearly indicate when you have done this.

Do not write in the table to the right.

1

[^0]1. Consider the following Laplace's equation in rectangle

$$
\left\{\begin{array}{l}
\Delta u=u_{x x}+u_{y y}=0, \quad 0<x<2, \quad 0<y<4 \\
u_{x}(0, y)=-2 \quad \text { and } \quad u_{x}(2, y)=0 \\
u_{y}(x, 0)=4 \quad \text { and } \quad u_{y}(x, 4)=0
\end{array}\right.
$$

and consider the following function

$$
u(x, y)=\frac{1}{2}\left(x^{2}-y^{2}\right)-2 x+4 y
$$

(a) (6 points) Verify that $u(x, y)$ satisfies the Laplace's equation $\Delta u=u_{x x}+u_{y y}=0$.

Solution: We need to compute $u_{x x}$ and $u_{y y}$;

$$
u_{x}=x-2 \quad \text { and } \quad u_{x x}=1
$$

Similarly,

$$
u_{y}=-y+4 \quad \text { and } \quad u_{y y}=-1
$$

Hence

$$
u_{x x}+u_{y y}=1-1=0
$$

Therefore, we conclude that given function $u(x, y)$ satisfies the Laplace's equation.
(b) (6 points) Verify also that $u(x, y)$ satisfies the given boundary conditions as well (check each of them separately) and conclude that $u(x, y)$ solves the above Laplace's equation with the given boundary conditions.
Solution: We need to check $u(x, y)$ satisfies all of the four boundary conditions; First boundary condition $u_{x}(0, y)=-2$;

$$
u_{x}(x, y)=x-2 \quad \text { therefore } \quad u_{x}(0, y)=0-2=-2
$$

Second boundary condition $u_{x}(2, y)=0$;

$$
u_{x}(x, y)=x-2 \quad \text { therefore } \quad u_{x}(2, y)=2-2=0
$$

Third boundary condition $u_{y}(x, 0)=4$;

$$
u_{y}(x, y)=-y+4 \quad \text { therefore } \quad u_{y}(x, 0)=-0+4=4
$$

Fourth boundary condition $u_{y}(x, 4)=0$;

$$
u_{y}(x, y)=-y+4 \quad \text { therefore } \quad u_{y}(x, 4)=-4+4=0
$$

Hence $u=u(x, y)$ satisfies all the boundary conditions. We then conclude that $u(x, y)$ is a solution to the Laplace's equation with the given boundary values.
2. Consider the following Heat conduction problem

$$
\left\{\begin{array}{l}
9 u_{x x}=u_{t}, \quad 0<x<\pi, \quad t>0 \\
u(0, t)=0 \quad \text { and } \quad u(\pi, t)=0 \\
u(x, 0)=3 \sin (3 x)+4 \sin (4 x)+10 \sin (10 x)
\end{array}\right.
$$

(a) (6 points) By considering separation of variables $u(x, t)=X(x) T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in $X$ and $T$ (take arbitrary constant as $-\lambda$ ).
Solution: Rewrite the PDE as $9 u_{x x}-u_{t}=0$. Let $u(x, t)=X(x) T(t)$. Then

$$
u_{x x}=X^{\prime \prime} T \quad \text { and } \quad u_{t}=X T^{\prime} .
$$

Substitute this in to the differential equation to get

$$
9 u_{x x}-u_{t}=9 X^{\prime \prime} T-X T^{\prime}=0 \quad \text { equivalently } \quad \frac{X^{\prime \prime}}{X}=\frac{T^{\prime}}{9 T}=-\lambda .
$$

Hence

$$
\begin{aligned}
& \frac{X^{\prime \prime}}{X}=-\lambda \quad \rightarrow \quad X^{\prime \prime}+\lambda X=0 \\
& \frac{T^{\prime}}{9 T}=-\lambda \quad \rightarrow \quad T^{\prime}+9 \lambda T=0
\end{aligned}
$$

(b) (5 points) Rewrite the boundary values in terms of $X$ and $T$.

Solution: We have at $x=0$

$$
u(0, t)=X(0) T(t)=0 ; \quad \text { one has either } \quad X(0)=0 \quad \text { or } \quad T(t)=0 .
$$

At $x=\pi$

$$
u(\pi, t)=X(\pi) T(t)=0 ; \quad \text { one has either } \quad X(\pi)=0 \quad \text { or } \quad T(t)=0 .
$$

(c) (5 points) Now choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to $X$.
Solution: We know that the choice of $T(t)=0$ gives only the trivial solution as $u(x, t)=$ $X(x) T(t)=0$.
Therefore, we choose our boundary conditions as $X(0)=0$ and $X(\pi)=0$ in order to obtain the non-trivial solution. Now if we rewrite the ordinary differential equation corresponding to $X$ we get

$$
X^{\prime \prime}+\lambda X=0, \quad X(0)=0 \quad \text { and } \quad X(\pi)=0
$$

(d) (5 points) Solve the two-point boundary value problem corresponding to $X$. Find all eigenvalues $\lambda_{n}$ and eigen-functions $X_{n}$
Solution: We know that the non-trivial general solutions is

$$
X(x)=A \cos (k x)+B \sin (k x)
$$

where we choose $\lambda=k^{2}, k>0$. (Other values of $\lambda, \lambda=0$ and $\lambda<0$, leads to trivial solutions again.). Using this general solution and the first boundary condition we gave

$$
X(0)=A \cos (0)+B \sin (0)=A=0
$$

Using the second boundary condition, we get (as $A=0$ )

$$
X(\pi)=B \sin (k \pi)=0
$$

This holds when $k \pi=n \pi$ for $n=1,2, \ldots$. Hence we get $k=n$, or equivalently, we get the eigenvalues

$$
\lambda_{n}=k^{2}=n^{2} \quad n=1,2, \ldots
$$

The corresponding eigenfunction (corresponding to $\lambda_{n}$ ) is

$$
X_{n}(x)=\sin (n x)
$$

(e) (5 points) For each eigenvalue $\lambda_{n}$ you found in (d), rewrite and solve the ordinary differential equation corresponding to $T_{n}$.
Solution: Since the ordinary differential equation corresponding to $T$ is

$$
T^{\prime}+9 \lambda T=0
$$

Plug in $\lambda=n^{2}$ we get (for each $n$ we have a different solution $T_{n}$ )

$$
T_{n}^{\prime}+9 n^{2} T_{n}=0
$$

We know that this is a first order linear ordinary differential equation and its solution is

$$
T_{n}(t)=C_{n} e^{-9 n^{2} t}
$$

for some $C_{n}$.
(f) (5 points) Now write general solution for each $n, u_{n}(x, t)=X_{n}(x) T_{n}(t)$ and find the general solution $u(x, t)=\sum u_{n}(x, t)$.
Solution: We know that the solution for each $n$ is

$$
u_{n}(x)=X_{n}(x) T_{n}(t)=\sin (n x) C_{n} e^{-9 n^{2} t}
$$

The general solution is

$$
u(x, t)=\sum_{n=1}^{\infty} u_{n}(x, t)=\sum_{n=1}^{\infty} C_{n} \sin (n x) e^{-9 n^{2} t}
$$

(g) (5 points) Using the given initial value and the general solution you found in (f), find the particular solution.
Solution: Now the initial value is given as $u(x, 0)=3 \sin (3 x)+4 \sin (4 x)+10 \sin (10 x)$. Hence, plug in $t=0$ in the solution we have in (f) gives us

$$
u(x, 0)=\sum_{n=1}^{\infty} C_{n} \sin (n x) e^{0}=\sum_{n=1}^{\infty} C_{n} \sin (n x)=3 \sin (3 x)+4 \sin (4 x)+10 \sin (10 x)
$$

From this we get that, all $C_{n}=0$ except $C_{3}=3, C_{4}=4$, and $C_{10}=10$. Hence in the solution, the only terms we have, for $n=2, n=4, n=10$ with $C_{3}=3, C_{4}=4$, and $C_{10}=10 ;$

$$
\begin{aligned}
u(x, t) & =\sum_{n=1}^{\infty} C_{n} \sin (n x) e^{-9 n^{2} t}=3 \sin (3 x) e^{-93^{2} t}+4 \sin (4 x) e^{-94^{2} t}+10 \sin (10 x) e^{-910^{2} t} \\
& =3 \sin (3 x) e^{-81 t}+4 \sin (4 x) e^{-144 t}+10 \sin (10 x) e^{-900 t}
\end{aligned}
$$

is the particular solution.
3. Consider the following wave equation which describes the displacement $u(x, t)$ of a piece of flexible string with the initial boundary value problem

$$
\left\{\begin{array}{l}
4 u_{x x}=u_{t t}, \quad 0<x<5, \quad t>0 \\
u(0, t)=0 \quad \text { and } \quad u(5, t)=0 \\
u(x, 0)=0 \quad \text { and } \quad u_{t}(x, 0)=\sin (\pi x) .
\end{array}\right.
$$

(a) (6 points) By considering separation of variables $u(x, t)=X(x) T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in $X$ and $T$.
Solution: Rewrite the differential equation as $4 u_{x x}-u_{t t}=0$. Let $u(x, t)=X(x) T(t)$. Then we get

$$
u_{x x}=X^{\prime \prime} T \quad \text { and } \quad u_{t t}=X T^{\prime \prime}
$$

Substitute this into the partial differential equation $4 u_{x x}-u_{t t}=0$ to get

$$
4 u_{x x}-u_{t t}=4 X^{\prime \prime} T-X T^{\prime \prime}=0
$$

Dividing by $4 X T$ we get

$$
\frac{X^{\prime \prime}}{X}=\frac{T^{\prime \prime}}{4 T}
$$

Notice that the left-hand side is a function of $x$ only and the right-hand side is function of $t$ only and as they are same, this is possible only if they are the same constant;

$$
\frac{X^{\prime \prime}}{X}=\frac{T^{\prime \prime}}{4 T}=-\lambda .
$$

From this we get

$$
\begin{aligned}
& \frac{X^{\prime \prime}}{X}=-\lambda \quad \rightarrow \quad X^{\prime \prime}+\lambda X=0 \\
& \frac{T^{\prime \prime}}{4 T}=-\lambda \quad \rightarrow \quad T^{\prime \prime}+4 \lambda T=0
\end{aligned}
$$

(b) (5 points) Rewrite the boundary values in terms of $X$ and $T$.

Solution: Since $u(x, t)=X(x) T(t)$ we have when $x=0$

$$
u(0, t)=X(0) T(t)=0 \quad \text { we should have either } \quad X(0)=0 \quad \text { or } \quad T(t)=0 .
$$

At $x=5$, we have

$$
u(5, t)=X(5) T(t)=0 \quad \text { we should have either } \quad X(5)=0 \quad \text { or } \quad T(t)=0
$$

(c) (5 points) Now choose the boundary values which will not give a non-trivial solution and then rewrite the ordinary differential equation with boundary values corresponding to $X$. Solution: We know that $T(t)=0$ will give only the trivial solution. Therefore, we should choose

$$
X(0)=0 \quad \text { and } \quad X(5)=0 .
$$

Then $X$ satisfies the following two-point boundary condition

$$
X^{\prime \prime}+\lambda X=0, \quad X(0)=0 \quad \text { and } \quad X(5)=0 .
$$

(d) (5 points) Solve the two-point boundary value problem corresponding to $X$ you found in (c). Find all eigenvalues $\lambda_{n}$ and eigenfunctions $X_{n}$

Solution: Now we know that only non-trivial solution comes from when $\lambda=k^{2}$ for some $k>0$ (again when $\lambda=0$ and $\lambda<0$ will give only the trivial solution, $X(x)=0$ ). In this case the solution is

$$
X(x)=A \cos (k x)+B \sin (k x) .
$$

We now find $A$ and $B$ using the boundary values we have in (c), at $x=0$

$$
X(0)=A \cos (0)+B \sin (0)=0 \quad \text { implies } \quad A=0 .
$$

At $x=5$, (now $A=0$, we only have $X(x)=B \sin (k x)$ )

$$
X(0)=B \sin (5 k)=0 .
$$

This holds if $\sin (5 k)=0$ which gives $5 k=\pi n$. Hence $k=\pi n / 5$.

$$
k=\frac{\pi n}{5} \quad \text { hence } \quad \lambda=k^{2}=\frac{\pi^{2} n^{2}}{5^{2}}
$$

Therefore, the eigenvalues are

$$
\lambda_{n}=\frac{\pi^{2} n^{2}}{5^{2}} .
$$

The corresponding eigenfunctions are

$$
X_{n}(x)=\sin (k x)=\sin \left(\frac{\pi n x}{5}\right)
$$

(e) (5 points) For each eaigenvalue $\lambda_{n}$ you found in (d), solve the initial value problem corresponding to $T_{n}$.
Solution: Since $\lambda_{n}=\frac{\pi^{2} n^{2}}{5^{2}}$ and the ordinary differential equation corresponding to $T$ is

$$
T^{\prime \prime}+4 \lambda T=0
$$

Substitute $\lambda=$ we get (we have a different solution for each $n$ );

$$
T_{n}^{\prime \prime}+4 \lambda T_{n}=T_{n}^{\prime \prime}+4 \frac{\pi^{2} n^{2}}{5^{2}} T_{n}=0
$$

This is a second order linear differential equation and which has characteristic equation (rewrite $4=2^{2}$ )

$$
r^{2}+\frac{2^{2} \pi^{2} n^{2}}{5^{2}}=0
$$

From this we get that the characteristic equation has a imaginary complex conjugate roots

$$
r= \pm \frac{2 \pi n}{5} \mathbf{i} .
$$

Thus the solutions are

$$
T_{n}(t)=A_{n} \cos \left(\frac{2 \pi n t}{5}\right)+B_{n} \sin \left(\frac{2 \pi n t}{5}\right) \quad n=1,2, \ldots,
$$

(f) (5 points) Now write general solution for each $n, u_{n}(x, t)=X_{n}(x) T_{n}(t)$ and find the general solution $u(x, t)=\sum u_{n}(x, t)$.
Solution: Combining (c) and (e) we have

$$
u_{n}(x, t)=X(x) T(t)=\sin \left(\frac{\pi n x}{5}\right)\left[A_{n} \cos \left(\frac{2 \pi n t}{5}\right)+B_{n} \sin \left(\frac{2 \pi n t}{5}\right)\right]
$$

The general solution is

$$
u(x, t)=\sum_{n=1}^{\infty} u_{n}(x, t)=\sum_{n=1}^{\infty} \sin \left(\frac{\pi n x}{5}\right)\left[A_{n} \cos \left(\frac{2 \pi n t}{5}\right)+B_{n} \sin \left(\frac{2 \pi n t}{5}\right)\right] .
$$

(g) (5 points) Using the given initial values, find the particular solution.

Solution: Now we have two initial values $u(x, 0)=0$ and $u_{t}(x, 0)=\sin (\pi x)$
At $t=0$, i.e, $u(x, 0)=0$

$$
u(x, 0)=\sum_{n=1}^{\infty} \sin \left(\frac{\pi n x}{5}\right)\left[A_{n} \cos (0)+B_{n} \sin (0)\right]=0
$$

gives us $A_{n}=0$ for every $n=1,2, \ldots$. Hence we have

$$
u(x, t)=\sum_{n=1}^{\infty} \sin \left(\frac{\pi n x}{5}\right) B_{n} \sin \left(\frac{2 \pi n t}{5}\right) .
$$

we now need to find $u_{t}(x, t)$;

$$
u_{t}(x, t)=\sum_{n=1}^{\infty} \sin \left(\frac{\pi n x}{5}\right) B_{n} \frac{2 \pi n}{5} \cos \left(\frac{2 \pi n t}{5}\right) .
$$

Then given initial value $u_{t}(x, 0)=\sin (\pi x)$ gives us

$$
u_{t}(x, 0)=\sum_{n=1}^{\infty} \sin \left(\frac{\pi n x}{5}\right) B_{n} \frac{2 \pi n}{5} \cos (0)=\sin (\pi x)
$$

Therefore, let $b_{n}=B_{n} \frac{2 \pi n}{5}$ then above identity becomes

$$
\sum_{n=1}^{\infty} b_{n} \sin \left(\frac{\pi n x}{5}\right)=\sin (\pi x)
$$

This tells us all $b_{n}=0$ except $b_{5}=1$ as

$$
\sum_{n=1}^{\infty} b_{n} \sin \left(\frac{\pi n x}{5}\right)=b_{1} \sin \left(\frac{\pi x}{5}\right)+b_{2} \sin \left(\frac{\pi 2 x}{5}\right)+\ldots b_{5} \sin \left(\frac{\pi 5 x}{5}\right)+\ldots
$$

and we know that this summation is $\sin (5 x)$ hence all $b_{n}$ should be zero except $b_{5}$ which will be 1 . Hence if we solve for $B_{5}$ here, (we plug in $n=5$ ) $1=b_{5}=B_{5} \frac{2 \pi 5}{5}$. Hence $B_{5}=1 /(2 \pi)$

$$
u(x, t)=B_{5} \sin \left(\frac{\pi 5 x}{5}\right) \sin \left(\frac{2 \pi 5 t}{5}\right)=\frac{1}{2 \pi} \sin (\pi x) \sin (2 \pi t)
$$

is the particular solution.
4. For the following differential equation

$$
4 x y^{\prime \prime}+2 y^{\prime}+y=0
$$

(a) (4 points) Find and classify all points as ordinary, regular singular, or irregular singular points.
Solution:

$$
y^{\prime \prime}+\frac{2}{4 x} y^{\prime}+\frac{1}{4 x} y=0 .
$$

Then $p(x)=2 / 4 x$ and $q(x)=1 / 4 x$ both have singularities at $x=0$. Therefore, all points except $x=0$ is ordinary points. For $x=0$ we need to check the following limits;

$$
\lim _{x \rightarrow 0} x p(x)=\lim _{x \rightarrow 0} \frac{2 x}{4 x}=1 / 2 \quad \text { and } \quad \lim _{x \rightarrow 0} x^{2} q(x)=\lim _{x \rightarrow 0} x^{2} \frac{1}{4 x}=0 .
$$

Since both limits exist and are finite, therefore $x=0$ is regular singular points.
(b) (5 points) For each of the regular point(s), find the corresponding indicial equation and find roots $r_{1}$ and $r_{2}$ of the indicial equation (Yes, there are two roots and the difference is not integer).
Solution: Since $x=0$ is the only regular singular point, we find the corresponding indicial equation. Let

$$
y(x)=x^{r} \sum_{n=0}^{\infty} a_{n} x^{n}=\sum_{n=0}^{\infty} a_{n} x^{n+r}
$$

be a solution for some $r$ and $a_{n}$. Find $y^{\prime}$ and $y^{\prime \prime}$ in terms of power series.

$$
y^{\prime}=\sum_{n=0}^{\infty}(n+r) a_{n} x^{n+r-1} \quad \text { and } \quad y^{\prime \prime}=\sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n+r-2} .
$$

Plug in to the differential equation to get

$$
4 x \sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n+r-2}+2 \sum_{n=0}^{\infty}(n+r) a_{n} x^{n+r-1}+\sum_{n=0}^{\infty} a_{n} x^{n+r}=0 .
$$

That is,

$$
\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_{n} x^{n+r-1}+\sum_{n=0}^{\infty} 2(n+r) a_{n} x^{n+r-1}+\sum_{n=0}^{\infty} a_{n} x^{n+r}=0
$$

In order to write the sums under one sum, we need to change the power of $x$ in the last summation from $n+r$ to $n+r-1$. Therefore,

$$
\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_{n} x^{n+r-1}+\sum_{n=0}^{\infty} 2(n+r) a_{n} x^{n+r-1}+\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}=0
$$

Now if we split $n=0$ terms in the first two sums and leave the rest under summation we get
$4 r(r-1) a_{0} x^{r-1}+\sum_{n=1}^{\infty} 4(n+r)(n+r-1) a_{n} x^{n+r-1}+2 r a_{0} x^{r-1}+\sum_{n=1}^{\infty} 2(n+r) a_{n} x^{n+r-1}+\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}=0$.

Now we have (combining $x^{r-1}$ and remaining terms under big sum)

$$
(4 r(r-1)+2 r) a_{0} x^{r-1}+\sum_{n=1}^{\infty}\left[4(n+r)(n+r-1) a_{n}+2(n+r) a_{n}+a_{n-1}\right] x^{n+r-1}=0
$$

Since this is true every $x$, we get

$$
\begin{equation*}
(4 r(r-1)+2 r) a_{0}=0 \quad \text { and } \quad 4(n+r)(n+r-1) a_{n}+2(n+r) a_{n}+a_{n-1}=0 \quad \text { for every } \quad n \geq 1 \tag{1}
\end{equation*}
$$

The first identity gives us (assuming $a_{0} \neq 0$ ) the indicial equation $4 r(r-1)+2 r=0$. Therefore, the roots are $r_{1}=1 / 2$ and $r_{2}=0$. We have two distinct roots and $r_{1}-r_{2}$ is not integer.
(c) (5 points) Find the corresponding recurrence relations for each of the roots $r_{1}$ and $r_{2}$.

Solution: Plug $r_{1}=1 / 2$ and solve for $a_{n}$ in (1) to get

$$
a_{n}=\frac{-a_{n-1}}{(2 n+1) 2 n}
$$

Plug in $r_{2}=0$ in (1) (this time i am writing the recurrence relation in terms of $b_{n}$ ) to get

$$
b_{n}=\frac{-b_{n-1}}{2 n(2 n-1)}
$$

(d) (5 points) Find the corresponding power series solution $y_{1}$ to $r_{1}$.

Solution: The recurrence relation for $r_{1}=1 / 2$ is $a_{n}=\frac{-a_{n-1}}{(2 n+1) 2 n}$ and if we check a few terms to find a pattern;

$$
\begin{aligned}
a_{1} & =\frac{-a_{0}}{32}=\frac{-a_{0}}{3!} \\
a_{2} & =\frac{-a_{1}}{54}=\frac{a_{0}}{543!}=\frac{a_{0}}{5!} \\
a_{3} & =\frac{-a_{2}}{76}=\frac{-a_{0}}{765!}=\frac{-a_{0}}{7!} \\
& \ldots \\
a_{n} & =(-1)^{n} \frac{a_{0}}{(2 n+1)!} \quad n=1,2, \ldots
\end{aligned}
$$

Therefore we get the first solution corresponding to $r_{1}=1 / 2$.

$$
y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n+\frac{1}{2}}=a_{0} \sum_{n=0}^{\infty}(-1)^{n} \frac{1}{(2 n+1)!} x^{n+\frac{1}{2}}
$$

(e) (5 points) Find the corresponding power series solution $y_{2}$ to $r_{2}$.

Solution: To find the solution corresponding to $r_{2}=0$, plug in $r_{2}=0$ in (1) (this time i am writing the recurrence relation in terms of $b_{n}$ ) to get

$$
b_{n}=\frac{-b_{n-1}}{2 n(2 n-1)}
$$

Now (assuming $b_{0}=a_{0} \neq 0$ )

$$
\begin{aligned}
b_{1} & =\frac{-b_{0}}{21}=\frac{-a_{0}}{2!} \\
b_{2} & =\frac{-b_{1}}{43}=\frac{b_{0}}{432!}=\frac{a_{0}}{4!} \\
b_{3} & =\frac{-b_{2}}{65}=\frac{-b_{0}}{654!}=\frac{-a_{0}}{6!} \\
& \ldots \\
b_{n} & =(-1)^{n} \frac{b_{0}}{(2 n)!} \quad n=1,2, \ldots,
\end{aligned}
$$

We get the second linearly independent solution corresponding to $r_{2}=0$

$$
y_{2}(x)=\sum_{n=0}^{\infty} b_{n} x^{n}=b_{0} \sum_{n=0}^{\infty}(-1)^{n} \frac{1}{(2 n)!} x^{n} .
$$

5. Let $f(x)$ be given as

$$
f(x)=x, \quad 0<x<2 .
$$

(a) (6 points) Extend $f(x)$ into an odd periodic function with period of 4.

Solution: As we want to extend $f$ into odd periodic function with period of 4 we define


Odd extension $F_{\text {odd }}(x)$ of $f$ is, $F_{\text {odd }}(x+2)=F_{\text {odd }}(x)$ and

$$
F_{\text {odd }}(x)=\left\{\begin{array}{cr}
f(x) & 0 \leq x<2, \\
-f(-x) & -2<x \leq 0 .
\end{array}=\left\{\begin{array}{lr}
x & 0 \leq x<2, \\
-(-x) & -2<x \leq 0
\end{array}\right.\right.
$$

(b) (6 points) Find Fourier series $F(x)$.

Solution: As we have the extension $F_{\text {odd }}$, which is periodic with period of $2 L=4$, we will find its Fourier series.
Since this is an odd extension, cosine terms will be zero; $a_{n}=0$ and $a_{0}=0$. Hence it will be a sine series; we only need to find $b_{n}$.

$$
\begin{aligned}
b_{n} & =\frac{1}{2} \int_{-2}^{2} f(x) \sin \left(\frac{n \pi x}{2}\right) d x \\
& =\frac{2}{2} \int_{0}^{2} x \sin \left(\frac{n \pi x}{2}\right) d x \\
& =\left.\frac{2}{n \pi}\left[-x \cos \left(\frac{n \pi x}{2}\right)\right]\right|_{x=0} ^{x=2}-\frac{2}{n \pi} \int_{0}^{2} \cos \left(\frac{n \pi x}{2}\right) d x \\
& =\frac{2}{n \pi}[-2 \cos (n \pi)]-\left.\frac{1}{2 n^{2} \pi^{2}} \sin (n \pi x)\right|_{x=0} ^{x=2} \\
& =\frac{-4}{n \pi}[\cos (n \pi)]-0=-\frac{(-1)^{n} 4}{n \pi} .
\end{aligned}
$$

Hence the Fourier series of $F_{\text {odd }}$ is

$$
\begin{aligned}
F(x) & =\frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos \left(\frac{n \pi x}{2}\right)+\sum_{n=1}^{\infty} b_{n} \sin \left(\frac{n \pi x}{2}\right) \\
& =0+0-\sum_{n=1}^{\infty} \frac{(-1)^{n} 4}{n \pi} \sin \left(\frac{n \pi x}{2}\right)
\end{aligned}
$$

6. (10 points (bonus)) Solve the first order equation

$$
3 u_{x}+2 u_{y}=0
$$

with the auxiliary condition

$$
u(x, 0)=\sin (x) .
$$

Solution:
Solution: Notice that

$$
\langle(3,2), \nabla u(x, y)\rangle=3 u_{x}+2 u_{y}=0
$$

Hence $u(x, y)$ is constant in the direction of $(3,2)$. The lines parallel to $(3,2)$ have equations $2 x-3 y=0$. here $2 x-3 y=0$ is called the characteristic lines. As $u(x, y)$ is constant on these lines therefore $u(x, y)$ depends only $2 x-3 y$. Hence

$$
u(x, y)=f(2 x-3 y)
$$

Using the the auxiliary condition $u(x, 0)=\sin (x)$ we get

$$
u(x, 0)=f(2 x)=\sin (x) .
$$

Since $f(2 x)=\sin (x)$ we can find $f(x)=\sin (x / 2)$. Since

$$
u(x, y)=f(2 x-3 y)=\sin ((2 x-3 y) / 2)
$$

is the solution. (you can check you answer by finding $u_{x}$ and $u_{y}$ and verify that $3 u_{x}+2 u_{y}=0$.)


[^0]:    ${ }^{1}$ Exam template credit: http://www-math.mit.edu/~psh

