UNIVERSITY OF CONNECTICUT

Spring 2018 - Math 3435
Exam 2 - March 28
Time Limit: 50 Minutes

Name (Print): ______________________

This exam contains 6 pages (including this cover page) an empty scratch paper and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	0	
Total:	40	

General solution to the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & 0 \le x \le 1,\ t \ge 0, \\ u(0,t) = 0,\ \ u(1,t) = 0, & t \ge 0, \\ u(x,0) = f(x). \end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

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[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. Let

$$f(x) = \begin{cases} 1 & \text{when } 0 \le x \le \pi, \\ 0 & \text{when } -\pi \le x < 0. \end{cases}$$

(a) (5 points) Find the Fourier series $\mathcal{F}(x)$ of $f(x)$ on $[-\pi, \pi]$

(b) (2 points) At which points on $[-\pi, \pi]$, do $\mathcal{F}(x)$ and $f(x)$ **NOT** agree ?

(c) (3 points) Verify that

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

2. (10 points) Solve

$$\begin{cases} u_t - u_{xx} = t\sin(2\pi x), & 0 \le x \le 1,\ t \ge 0, \\ u(0,t) = 0,\ \ u(1,t) = 0, & t \ge 0, \\ u(x,0) = \sin(3\pi x) & 0 \le x \le 1. \end{cases} \tag{1}$$

3. (10 points) Consider the Heat equation

$$\begin{cases} u_t - 4u_{xx} = \sin(2\pi x)t + 2xt, \quad 0 \le x \le 1, \ t \ge 0, \\ u(0,t) = \mathbf{1}, \ \ u_x(1,t) = \mathbf{t^2}, \\ u(x,0) = 1 + \sin(3\pi x) - x. \end{cases} \tag{2}$$

where the boundary conditions are **non-homogeneous**. Transform the equation into a new one with **homogeneous** boundary conditions. (You do not need to solve the new equation).

4. (10 points) Describe the steps how to solve the following heat equation

$$\begin{cases} u_t - ku_{xx} = h(x,t), & 0 \le x \le \pi,\ t \ge 0, \\ u(0,t) = a(t),\ \ u(\pi,t) = b(t), & \\ u(x,0) = f(x). & \end{cases} \tag{3}$$

5. (10 points (bonus)) For a given ϕ with $\lim_{x\to\pm\infty}\phi(x) = 0$ "rapidly", the solution to the following Heat conduction

$$\begin{cases} u_t = ku_{xx}, & -\infty \leq x \leq \infty,\ t \geq 0, \\ u(x,0) = \phi(x) & -\infty \leq x \leq \infty \end{cases} \tag{4}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} exp\left(-\frac{(y-x)^2}{4kt}\right) dx.$$

Here $k > 0$.

(a) Using (4) show that $C(t) = \int_{-\infty}^{\infty} u(x,t)dx$ remains constant in time. You may assume $\lim_{x\to\pm\infty} u_x(x,t) = 0$. (Hint: Use (4) and do integration by parts to show that $C'(t) = 0$ for every $t \geq 0$).

(b) Using (4) show that $E(t) = \int_{-\infty}^{\infty} u^2(x,t)dx$ decreases in time. ((Hint: Use (4) and do integration by parts to show that $E'(t) < 0$ for every $t \geq 0$).