General solution to the heat equation

$$\begin{cases} 
  u_t - u_{xx} = 0, & 0 \leq x \leq 1, \ t \geq 0, \\
  u(0, t) = 0, \ u(1, t) = 0, & t \geq 0, \\
  u(x, 0) = f(x).
\end{cases}$$

is

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 t} \sin(n\pi x).$$
1. Let
\[ f(x) = \begin{cases} 1 & \text{when } 0 \leq x \leq \pi, \\ 0 & \text{when } -\pi \leq x < 0. \end{cases} \]

(a) (5 points) Find the Fourier series \( F(x) \) of \( f(x) \) on \([−π, π]\)

(b) (2 points) At which points on \([−π, π]\), do \( F(x) \) and \( f(x) \) NOT agree?

(c) (3 points) Verify that
\[ \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}. \]
2. (10 points) Solve

\[
\begin{cases}
  u_t - u_{xx} = t \sin(2\pi x), & 0 \leq x \leq 1, \ t \geq 0, \\
  u(0,t) = 0, \ u(1,t) = 0, & t \geq 0, \\
  u(x,0) = \sin(3\pi x) & 0 \leq x \leq 1.
\end{cases}
\]
3. (10 points) Consider the Heat equation

\[
\begin{cases}
    u_t - 4u_{xx} = \sin(2\pi x)t + 2xt, & 0 \leq x \leq 1, \ t \geq 0, \\
    u(0, t) = 1, \ u_x(1, t) = t^2, \\
    u(x, 0) = 1 + \sin(3\pi x) - x.
\end{cases}
\] (2)

where the boundary conditions are non-homogeneous. Transform the equation into a new one with homogeneous boundary conditions. (You do not need to solve the new equation).
4. (10 points) Describe the steps how to solve the following heat equation

\[
\begin{align*}
    u_t - ku_{xx} &= h(x, t), \\
    0 &\leq x \leq \pi, \ t \geq 0, \\
    u(0, t) &= a(t), \ u(\pi, t) = b(t), \\
    u(x, 0) &= f(x).
\end{align*}
\]
5. (10 points (bonus)) For a given \( \phi \) with \( \lim_{x \to \pm \infty} \phi(x) = 0 \) “rapidly”, the solution to the following Heat conduction

\[
\begin{align*}
  u_t &= ku_{xx}, \quad -\infty \leq x \leq \infty, \quad t \geq 0, \\
  u(x, 0) &= \phi(x) \quad -\infty \leq x \leq \infty
\end{align*}
\]

is given by

\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp \left( -\frac{(y-x)^2}{4kt} \right) \, dx.
\]

Here \( k > 0 \).

(a) Using (4) show that \( C(t) = \int_{-\infty}^{\infty} u(x, t) \, dx \) remains constant in time. You may assume \( \lim_{x \to \pm \infty} u_x(x, t) = 0 \). (Hint: Use (4) and do integration by parts to show that \( C'(t) = 0 \) for every \( t \geq 0 \)).

(b) Using (4) show that \( E(t) = \int_{-\infty}^{\infty} u^2(x, t) \, dx \) decreases in time. ((Hint: Use (4) and do integration by parts to show that \( E'(t) < 0 \) for every \( t \geq 0 \)).