UCONN
UNIVERSITY OF CONNECTICUT

Spring 2018 - Math 3435
Final Exam - April 30
Time Limit: 120 Minutes

Name (Print): ______________________

This exam contains 10 pages (including this cover page) an empty scratch paper and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.
You may find the following identities useful in Question 5. The Laplace's equation in polar coordinates is

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0$$

and the general solution to this is

$$U(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

Also the following identity might be useful in the same question

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).$$

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	0	
Total:	80	

[1]

[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. (10 points) Use the method of characteristics the solve the first-order partial differential equation for $u = u(x, y)$

$$u_x - \frac{e^x}{1 + e^y} u_y = 0 \quad \text{for } -\infty < x < \infty,\ y > 0$$

satisfying the side condition $u(x, 0) = e^{2x}$.

2. Let $f(x)$ be given as

$$f(x) = x \quad \text{when} \quad 0 \leq x < 1.$$

(a) (2 points) Extend $f(x)$ into an odd periodic function with period of 2.

(b) (4 points) Find Fourier series $F(x)$ of the function you found in (a).

(c) (4 points) Using part (a)-(b), verify that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

3. (10 points) Solve the Poisson equation

$$\begin{cases} u_{xx} + u_{yy} = x^2 + y^2, & \text{when } x^2 + y^2 < 1, \\ u(x, y) = 0, & \text{when } x^2 + y^2 = 1. \end{cases}$$

using the polar coordinates. You may want to look for solutions of the form $u(x, y) = f(x^2 + y^2)$.

4. (10 points) Suppose that$u(x,t)$ is solution of the diffusion equation with variable dissipation

$$\begin{cases} u_t - ku_{xx} + h(t)u = 0 & \text{when } -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x) & \text{when } -\infty < x < \infty. \end{cases}$$

and $g(t)$ is a solution to $g'(t) = h(t)g(t)$ with $g(0) = 1$. Then show that $v(x,t) = g(t)u(x,t)$ is a solution of

$$\begin{cases} v_t - kv_{xx} = 0 & \text{when } -\infty < x < \infty,\ t > 0, \\ v(x,0) = f(x) & \text{when } -\infty < x < \infty. \end{cases}$$

5. Let $u(x, y)$ be the solution to the following Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{when } x^2 + y^2 < 1, \\ u(x, y) = 4x^3 & \text{when } x^2 + y^2 = 1 \end{cases}$$

You may want to rewrite the Dirichlet problem in polar coordinates (including the boundary condition).

(a) (5 points) Find the solution $u(x, y)$. [See the cover page for the general solution and a hint].

(b) (2 points) Rewrite the solution you found in (a) in Cartesian coordinates, i.e. (x, y), and verify that it is the solution of the Laplace equation satisfying the given boundary condition.

(c) (3 points) Find the maximum value of $u(x, y)$ in the disk of radius 1.

6. (10 points) Find the solution $u(x, t)$ to the following Wave equation

$$\begin{cases} u_{tt} - a^2 u_{xx} = e^{-t}\cos(x) & \text{when } -\infty < x < \infty,\ -\infty < t < \infty, \\ u(x,0) = \sin(x) \text{ and } u_t(x,0) = a\cos(x), & \text{when } -\infty < x < \infty. \end{cases}$$

7. Let $u(x,t)$ be the solution to the following heat equation

$$\begin{cases} u_t - u_{xx} = \frac{1}{\pi}xe^t + t[2 - \frac{2}{\pi}x + \sin(x)] & \text{when } 0 \leq x \leq \pi,\ t \geq 0 \\ u(0,t) = t^2 \text{ and } u(\pi,t) = e^t & \text{when } t \geq 0, \\ u(x,0) = \frac{x}{\pi} + \sin(2x) & \text{when } 0 \leq x \leq \pi. \end{cases}$$

(a) (3 points) Find a particular solution u_p and let $v(x,t) = u(x,t) - u_p(x,t)$ so that $v(x,t)$ satisfies the homogeneous boundary condition and solves the non-homogeneous heat equation.

(b) (3 points) Write the PDE for which $v(x,t)$ solves, the boundary conditions and the initial condition $v(x,t)$ satisfies.

(c) (4 points) Without solving the new equation corresponding to v describe how to solve it.

8. Assume that $u(x,t)$ satisfies the following diffusion equation

$$\begin{cases} u_{tt} = u_{xx} & \text{when } -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) & \text{when } -\infty < x < \infty \end{cases}$$

where $f, g \in C^2$. You can assume that $f(x)$ and $g(x)$ vanish when $|x|$ is big, say $|x| > 10^{3435}$. Define

$$F(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx \quad \text{and} \quad G(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx.$$

(a) (3 points) Show that $E(t) = F(t) + G(t)$ is constant for $t \geq 0$ (You may use $u_x(x,t) \to 0$ as $x \to \pm\infty$).

(b) (3 points) Compute $E(0)$. (You may use $u_x(x,0) = f'(x)$.)

(c) (4 points) Suppose that $E(t) = 0$ for every $t \geq 0$. Show that $u(x,t) =$ constant.

9. (10 points (bonus)) Use parts (a)-(c) in Question 8 to show that the Dirichlet problem in Question 8

$$\begin{cases} u_{tt} = u_{xx} & \text{when } -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) & \text{when } -\infty < x < \infty \end{cases}$$

has at most one solution. Here $f, g \in C^2$ and $f(x)$ and $g(x)$ vanish when $|x|$ is big, say $|x| > 10^{3435}$.