UCONN - Math 3435 - Spring 2018 - Problem set 1

Question 1 (Exercise 1.2, 1b) *Show that u satisfies the given PDE*

$$u(x,y) = f(x) + g(y); \quad u_{xy} = 0 \text{ where the functions } f \text{ and } g \text{ are assumed to be } C^2.$$

Solution: Since $f, g \in C^2$ we can take first x derivative and then y derivative of u

$$u_x = f'(x) \quad \text{then} \quad u_{xy} = 0.$$

Hence u satisfies the given PDE.

Question 2 (Exercise 1.2, 1d) *Show that u satisfies the given PDE*

$$u(x,t) = x^2 + 2t \quad \text{and} \quad u_{xx} = u_t.$$

Solution: Since

$$u_x = 2x \; u_{xx} = 2 \quad \text{and} \quad u_t = 2$$

we get $u_{xx} = u_t$. Hence u satisfies the given PDE.

Question 3 (Exercise 1.2, 1f) *Show that u satisfies the given PDE*

$$u(x,t) = \sin(x - ct) \quad u_{tt} - c^2 u_{xx} = 0 \quad \text{where } c \text{ is real constant}.$$

Solution: Since

$$u_x = \cos(x - ct) \quad \text{and} \quad u_{xx} = -\sin(x - ct)$$

and

$$u_t = -c\cos(x - ct) \quad \text{and} \quad u_{tt} = -c^2 \sin(x - ct).$$

Combining these two we get

$$-c^2 \sin(x - ct) - (-c^2 \sin(x - ct)) = 0.$$

Hence u satisfies the given PDE.

Question 4 (Exercise 1.2, 2c) *Show that* $\log(x^2 + y^2)$, $x^2 + y^2 \neq 0$ *is a solution of Laplace's equation* $u_{xx} + u_{yy} = 0$.

Solution: Since

$$u_x = \frac{2x}{x^2 + y^2} \quad \text{and} \quad u_{xx} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}$$

Similarly,

$$u_y = \frac{2y}{x^2 + y^2} \quad \text{and} \quad u_{yy} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2}.$$

Combining these two we get

$$\begin{aligned} u_{xx} + u_{yy} &= \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \\ &= \frac{4}{x^2 + y^2} - \frac{4x^2 + 4y^2}{(x^2 + y^2)^2} \\ &= \frac{4}{x^2 + y^2} - \frac{4}{x^2 + y^2} = 0. \end{aligned}$$

Hence u satisfies Laplace's equation.

Question 5 (Exercise 1.2, 3d) *Show that the given function u solves the heat equation* $u_t - ku_{xx} = 0$ *where*

$$u(x,t) = \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right).$$

Solution: Since

$$u_t = -\frac{1}{2}\frac{1}{\sqrt{k}}\frac{1}{t^{3/2}} \exp\left(\frac{-x^2}{4kt}\right) + \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right)\frac{x^2}{4kt^2}$$

and

$$u_x = \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right)\left(\frac{-2x}{4kt}\right) \quad \text{and} \quad u_{xx} = \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right)\left(\frac{-2}{4kt}\right) + \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right)\left(\frac{-2x}{4kt}\right)\left(\frac{-2x}{4kt}\right).$$

Combining these all together we see that (after some algebra)

$$\begin{aligned} u_t - ku_{xx} &= -\frac{1}{2}\frac{1}{\sqrt{k}}\frac{1}{t^{3/2}} \exp\left(\frac{-x^2}{4kt}\right) + \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right)\frac{x^2}{4kt^2} \\ &\quad - k\left(-\frac{1}{2}\frac{1}{\sqrt{k}}\frac{1}{t^{3/2}} \exp\left(\frac{-x^2}{4kt}\right) + \frac{1}{\sqrt{kt}} \exp\left(\frac{-x^2}{4kt}\right)\frac{x^2}{4kt^2}\right) \\ &= 0. \end{aligned}$$

Question 6 (Exercise 1.2, 4d) *Show that the given function u satisfies the wave equation* $u_{tt} - c^2u_{xx} = 0$ *for some c where*

$$u(x,t) = f(x+2t) + g(x-2t) \quad \textit{for } f, g \in C^2.$$

Solution: Notice that f, g are function of single variable and twice differentiable and therefore

$$u_x = f'(x+2t) + g'(x-2t) \quad \text{and} \quad u_{xx} = f''(x+2t) + g''(x-2t).$$

Similarly,

$$u_t = 2f'(x+2t) - 2g'(x-2t) \quad \text{and} \quad u_{tt} = 4f''(x+2t) + 4g''(x-2t).$$

Hence

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 4f''(x+2t) + 4g''(x-2t) - c^2(f''(x+2t) + g''(x-2t)) \\ &= (4-c^2)(f''(x+2t) + g''(x-2t)). \end{aligned}$$

Hence u satisfies the wave equation $u_{tt} - c^2u_{xx} = 0$ iff $c = \pm 2$.

Question 7 (Exercise 1.2, 5c and 5d) *Write the order of the given PDE and classify them as linear or nonlinear. If the PDE is linear, specify if it is homogeneous or non-homogeneous.*

$$5c)\ u_{xxyy} + e^x u_x = y \quad \textit{and} \quad 5d)\ uu_{xx} + u_{yy} - u = 0.$$

Solution: $u_{xxyy} + e^x u_x = y$ is a fourth order linear PDE and it is non-homogeneous as the right-hand-side contains a nonzero function y which does not contain derivative of u. On the other hand, $uu_{xx} + u_{yy} - u = 0$ is a second order nonlinear PDE due the term uu_{xx}.

Question 8 (Exercise 1.2, 12) *For what values of the positive constants m and n the PDE*

$$u_{xx} + u_{yy} + mu_{xy} + u_x + nu_y = 0$$

be (a) hyperbolic, (b) elliptic, (c) parabolic.

Solution: By the Classification theorem at page 31, we have to look for the sign of $b^2 - 4ac$ where in this problem $b = m$, $a = 1$ and $c = 1, d = 1, e = n, k = 0$. Hence we get

$$b^2 - 4ac = m^2 - 4.$$

The PDE is hyperbolic if $b^2 - 4ac = m^2 - 4 > 0$. Therefore, when $m > 2$ or $m < -2$ the PDE is hyperbolic.

The PDE is elliptic if $b^2 - 4ac = m^2 - 4 < 0$. Therefore, when $-2 < m < 2$ the PDE is elliptic.

The PDE is parabolic if $b^2 - 4ac = m^2 - 4 = 0$ and $2cd \neq de$ or $2ae \neq bd$. Now if $m = \pm 2$ and since $2cd = 2$, $be = mn$ hence we should $2 \neq mn$ or since $2ae = 2n$ and $bd = m$ hence we should have $2n \neq m$. Hence if $m = \pm 2$ and either $n \neq 2/m = \pm 1$ or $n \neq m/2 = \pm 1$ then the PDE is parabolic.

Question 9 (Exercise 1.2, 13) *It is given that $u_1(x,y) = x^2$ solves $u_{xx} + u_{yy} = 2$ and $u_2(x,y) = cx^3 + dy^3$ solves $u_{xx} + u_{yy} = 6cx + 6dy$ for some real constants c and d.*
(a) Find a solution of $u_{xx} + u_{yy} = Ax + By + C$ for given A, B, C.
(b) How can many more solutions of the problem (c) be produced?

Solution: (a) Notice that $u_{xx} + u_{yy} = 0$ is a linear PDE. Therefore, any linear combinations of solutions will also be solutions. Consider $u(x,y) = mu_1(x,y) + nu_2(x,y)$ where m, n are constants to be chosen so that u satisfies $u_{xx} + u_{yy} = Ax + By + C$. Since $mu_1(x,y)$ solves $u_{xx} + u_{yy} = 2m$ and $nu_2(x,y)$ solves $u_{xx} + u_{yy} = 6cx + 6dy$. Then $u(x,y) = mu_1(x,y) + nu_2(x,y)$ solves $u_{xx} + u_{yy} = 2m + n(6cx + 6dy)$ and we want the right-hand side to be $Ax + By + C$. Therefore, $2m = C$, $6cxn = Ax$, and $6dyn = By$. If we choose $m = C/2$, $cn = A/6$ and $dn = B/6$ then

$$u(x,y) = mu_1(x,y) + nu_2(x,y) = C/2x^2 + A/6x^3 + B/6y^3$$

solves $u_{xx} + u_{yy} = Ax + By + C$.

(b) We want to generate more solutions by using the linearity of the above PDE. If we can find any solution to homogeneous equation $u_{xx} + u_{yy} = 0$, for example $u_3(x,y) = x^2 - y^2$. Then the solution to non-homogeneous part $u(x,y) = C/2x^2 + A/6x^3 + B/6y^3$ plus any constant multiple of solution to homogeneous part will give us a new solution to $u_{xx} + u_{yy} = Ax + By + C$. Hence

$$u(x,y) + Du_3(x,y) = \frac{C}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{6}y^3 + D(x^2 - y^2)$$

solves $u_{xx} + u_{yy} = Ax + By + C$ for given A, B, C for arbitrary constant D so that we can generate infinitely many solutions.

Question 10 (Exercise 1.3, 1b) *Find the general solution to $u_{xy} = x^2 y$ where $u = u(x,y)$.*

Solution: Since $u_{xy} = x^2 y$ we can integrate first with respect to x and then y;

$$u_x = \int u_{xy} dy = \int x^2 y dy = x^2 \frac{y^2}{2} + f(x)$$

where $f(x)$ is some C^1 function. If we integrate now with respect to x we get

$$u(x,y) = \int u_x dx = \int (x^2 \frac{y^2}{2} + f(x)) dx = \frac{x^3}{3}\frac{y^2}{2} + \int f(x) dx + g(y)$$

where $g(y)$ is a C^2 function.

Question 11 (Exercise 1.3, 1c) *Find the general solution to $u_{xyz} = 0$ where $u = u(x,y,z)$.*

Solution: First integration with respect to z gives

$$u_{xy} = \int u_{xyz} dz = f(x,y)$$

where $f(x,y)$ is a C^1 function. Then if we integrate with respect to y we get

$$u_x = \int u_{xy} dy = \int f(x,y) dy + g(x,z)$$

where $g(x,z)$ is a C^2 function. Finally, if we integrate with respect to x we get

$$u(x,y,z) = \int u_x dx = \iint f(x,y) dy dx + \int g(x,z) dx + h(y,z)$$

where $h(y,z)$ is a C^3 function. Or we can write these integrals as some functions as

$$F(x,y) = \iint f(x,y) dy dx \quad \text{and} \quad G(x,z) = \int g(x,z).$$

Then $u(x,y,z) = F(x,y) + G(x,y) + h(y,z)$ where F, G, h are C^3 functions.