Question 1 Consider the following Dirichlet problem

\[
\begin{cases}
U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 & \text{in } B(0, 2), \\
U(2, \theta) = 1 + 3 \sin(2) & .
\end{cases}
\]

1. Using the Poisson’s integral formula write down solution to the above Dirichlet problem.

2. Find the maximum value of \( U(r, \theta) \) on disk with radius 2.

3. Using the Poisson integral formula you found in the first part, find the value of \( U \) at the origin.

Solution: 1. We know that the solution to Dirichlet problem in a ball of radius \( a \) centered at origin with boundary value \( f(\theta) \) is

\[
U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 + 2ar \cos(\theta - s) + r^2} f(s) ds.
\]

Hence as \( a = 2 \) and \( f(\theta) = 1 + 3 \sin(2) \) we have

\[
U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{2^2 - r^2}{2^2 + 4r \cos(\theta - s) + r^2} [1 + 3 \sin(2s)] ds.
\]

2. We next find the maximum of \( U \). Since \( U(r, \theta) \) is harmonic in \( B(0, 2) \) then it attains it maximum value on the boundary of \( B(0, 2) \). That is

\[
\max_{\partial B(0, 2)} U = \max_{\partial B(0, 2)} U(r, \theta) = \max U(2, \theta) = \max [1 + 3 \sin(2\theta)] = 4.
\]

3. Using part 1. we want to find the value of \( U \) at the origin. Now in the Cartesian coordinate system origin \( = (0, 0) = (x, y) \). We know that \( r^2 = x^2 + y^2 = 0 \). Hence we evaluate the Poisson integral formula at \( r = 0 \),

\[
U(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{2^2 - 0^2}{2^2 + 4 \times 0 \times \cos(\theta - s) + 0^2} [1 + 3 \sin(2s)] ds
\]

\[
= \frac{1}{2\pi} \int_0^{\infty} [1 + 3 \sin(2s)] ds
\]

\[
= \frac{1}{2\pi} \left[ s - \frac{3}{2} \cos(2s) \right] = 1.
\]

Hence \( U(0, \theta) = 1 \).