UCONN - Math 3435 - Spring 2018 - Problem set 10

Question 1 *Consider the following Dirichlet problem*

$$\begin{cases} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0 & in\ B(0,2), \\ U(2,\theta) = 1 + 3\sin(2) & . \end{cases}$$

1. *Using the Poisson's integral formula write down solution to the above Dirichlet problem.*
2. *Find the maximum value of $U(r,\theta)$ on disk with radius 2.*
3. *Using the Poisson integral formula you found in the first part, find the value of U at the origin.*

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Solution: 1. We know that the solution to Dirichlet problem in a ball of radius a centered at origin with boundary value $f(\theta)$ is

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 + 2ar\cos(\theta - s) + r^2} f(s)ds.$$

Hence as $a = 2$ and $f(\theta) = 1 + 3\sin(2)$ we have

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{2^2 - r^2}{2^2 + 4r\cos(\theta - s) + r^2}[1 + 3\sin(2s)]ds.$$

2. We next find the maximum of U. Since $U(r,\theta)$ is harmonic in $B(0,2)$ then it attains it maximum value on the boundary of $B(0,2)$. That is

$$\max_{\bar{B}(0,2)} U(R,\theta) = \max_{\partial B(0,2)} U(r,\theta) = \max U(2,\theta) = \max[1 + 3\sin(2\theta)] = 4.$$

3. Using part 1. we want to find the value of U at the origin. Now in the Cartesian coordinate system origin= $(0,0) = (x,y)$. We know that $r^2 = x^2 + y^2 = 0$. Hence we evaluate the Poisson integral formula at $r = 0$,

$$\begin{aligned} U(0,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{2^2 - 0^2}{2^2 + 4 \times 0 \times \cos(\theta - s) + 0^2}[1 + 3\sin(2s)]ds \\ &= \frac{1}{2\pi}\int_0^{\infty} [1 + 3\sin(2s)]ds \\ &= \frac{1}{2\pi}[s - \frac{3}{2}\cos(2s)] = 1. \end{aligned}$$

Hence $U(0,\theta) = 1$.