UCONN - Math 3435 - Spring 2018 - Problem set 2

Question 1 (Exercise 1.3, 2c) *Find general solution of the following PDE for $u = u(x,y)$ using ODE techniques.*

$$u_x + 2xu = 4xy.$$

Solution: Here we should multiply the PDE with the integrating factor $\mu(x) = e^{\int 2xdx} = e^{x^2}$ to get

$$e^{x^2}u_x + e^{x^2}2xu = e^{x^2}4xy$$

Then the left hand side can be written as

$$(e^{x^2}u)_x = e^{x^2}4xy.$$

Hence we now can integrate both sides with respect to x to get

$$\begin{aligned} e^{x^2}u(x,y) = \int (e^{x^2}u)_x dx &= \int e^{x^2}4xydx \\ &= 2y\int e^{x^2}2xdx = 2ye^{x^2} + f(y) \end{aligned}$$

for some $f(y) \in C^1$. Hence general solution is

$$u(x,y) = 2y + \frac{f(y)}{e^{x^2}} \quad \text{for some } f \in C^1.$$

Question 2 (Exercise 1.3, 2d) *Find general solution of the following PDE for $u = u(x,y)$ using ODE techniques.*

$$yu_{xy} + 2u_x = x.$$

Solution: As hint suggested we integrate with respect to x first to get

$$\int yu_{xy}dx + \int 2u_x dx = \int xdx$$

we get

$$yu_y + 2u = \frac{x^2}{2} + f(y)$$

for some $f \in C^1$. To find the integrating factor we divide everyting by y

$$u_y + \frac{2}{y}u = \frac{x^2}{2y} + \frac{f(y)}{y}$$

Since integrating factor is $\mu(y) = e^{\int \frac{2}{y}dy} = y^2$, then we multiply the PDE with the integrating factor to get

$$y^2u_y + 2yu = \frac{x^2y}{2} + yf(y)$$

for some $f(y) \in C^1$ The left hand side can be written as

$$(y^2u)_y = \frac{x^2y}{2} + yf(y).$$

We integrate both sides with respect to y to get

$$y^2u(x,y) = \int (y^2u)_y dy = \int \frac{x^2y}{2}dy + \int yf(y)dy$$
$$= \frac{x^2y^2}{4} + \int yf(y)dy + g(x)$$

for some $g \in C^1$. Hence general solution is

$$u(x,y) = \frac{x^2}{4} + \frac{1}{y^2}\int yf(y)dy + \frac{g(x)}{y^2}$$

for some $f, g \in C^1$. Since $\int yf(y)dy = F(y)$ then we can write it as

$$u(x,y) = \frac{x^2}{4} + \frac{1}{y^2}F(y) + \frac{g(x)}{y^2}$$

Question 3 (Exercise 1.3, 3c) *For PDE in Problem 2c, find a particular solution satisfying*

$$u(x,x) = 0, \text{ i.e., } u = 0 \text{ on } y = x.$$

Solution: Since general solution in Problem 2c is

$$u(x,y) = 2y + \frac{f(y)}{e^{x^2}} \quad \text{for some } f \in C^1.$$

We evaluate at $y = x$ to get

$$0 = u(x,x) = 2x + \frac{f(x)}{e^{x^2}}.$$

If we solve $f(x)$ to get

$$f(x) = -2xe^{x^2}.$$

Hence the particular solution is

$$u(x,y) = 2y + \frac{f(y)}{e^{x^2}} = 2y + \frac{-2ye^{y^2}}{e^{x^2}} = 2y(1 - e^{y^2-x^2}).$$

Question 4 (Exercise 1.3, 3d) *For PDE in Problem 2d, find a particular solution satisfying*

$$u(x,1) = 0 \quad \text{and} \quad u(0,y) = 0.$$

Solution: Since general solution in Problem 2d is

$$u(x,y) = \frac{x^2}{4} + \frac{1}{y^2}F(y) + \frac{g(x)}{y^2}.$$

we first use $u(x,1) = 0$ to get

$$0 = u(x,1) = \frac{x^2}{4} + \frac{1}{1^2}F(1) + \frac{g(x)}{1^2}.$$

If we solve for $g(x)$ we get

$$g(x) = -\frac{x^2}{4} - F(1)$$

If we use second condition $u(0,y) = 0$ and substitute $g(x)$ in the solution we get

$$0 = u(0,y) = 0 + \frac{1}{y^2}F(y) + \frac{-\frac{0^2}{4} - F(1)}{y^2} =$$

If we solve for $F(y)$ we get

$$F(y) = F(1).$$

which tells us that $F(y) = c$ for some c. Now we substitute everything in u and get

$$u(x,y) = \frac{x^2}{4} + c\frac{1}{y^2} + \frac{-\frac{x^2}{4} - c}{y^2} = \frac{x^2}{4} - \frac{x^2}{4y^2} = \frac{x^2}{4}(1 - \frac{1}{y^2}).$$

Question 5 (Exercise 1.3, 8b) *Find some constants a, b such that $u(x,y) = f(ax+by)$ is a general solution to $5u_x + 6u_y$ where $f \in C^1$ some arbitrary function where*

Solution: Since we are looking for a, b where $u(x,y) = f(ax+by)$ is a solution to $5u_x + 6u_y$ we need to find u_x, u_y and substitute in the PDE and find a, b. Notice that f is a one variable function hence

$$u_x = af'(ax+by) \quad \text{and} \quad u_y = bf'(ax+by).$$

If we substitute this into the PDE we get

$$5u_x + 6u_y = 5af'(ax+by) + 6bf'(ax+by) = 0$$

That is, $(5a+6b)f'(ax+by) = 0$ or $b = -5a/6$ (since f is arbitrary therefore $f' \not\equiv 0$). Hence $u(x,y) = f(ax - 5a/6y)$ is a solution to above PDE for arbitrary constant a and function $f \in C^1$. For example we can take $a = 6$ and $b = -1$ to get $u(x,y) = f(6x - y)$ as solution to the PDE.

Question 6 (Exercise 1.3, 9c) *Use the technique in Problem 8 to solve the following PDE*

$$3u_x - 4u_y = 0 \quad \textit{with} \quad u(x,x) = x^2 - x.$$

Solution: We are looking for a solution $u(x,y) = f(ax+by)$ and need to find a, b first. Since u is solution to $3u_x - 4u_y = 0$ we then get

$$u_x = af'(ax+by) \quad \text{and} \quad u_y = bf'(ax+by)$$

we plug in to the PDE to get

$$0 = 3u_x - 4u_y = 3af''(ax+by) - 4bf'(ax+by)$$

which gives us $3a - 4b = 0$ or $b = 3a/4$. For simplicity, choose $a = 4$ and then $b = 3$. Hence $u(x,y) = f(4x+3y)$. Now using the given condition

$$x^2 - x = u(x,x) = f(4x+3x).$$

We want to find $f(x)$ therefore, substitute $z = 7x$ to get $f(z) = f(7x) = x^2 - x = (z/7)^2 - z$. Hence $f(z) = (z/7)^2 - z$ and the solution is

$$u(x,y) = f(4x+3y) = ((4x+3y)/7)^2 - (4x+3y).$$