UCONN - Math 3435 - Spring 2018 - Problem set 3

Question 1 (Exercise 2.1, 1c) *Find the general solution of the following PDE where* $u = u(x, y)$ *and*

$$1u_x + 2u_y - 4u = e^{x+y}.$$

Solution: We use the idea developed in section 2.1; and do the following change of variables;

$$\begin{cases} w = 2x - 1y, & x = \frac{1}{2}(w + z) \\ z = y, & y = z \end{cases}$$

We look for $v(w, z) = u(x, y) = u(1/2(w + z), z)$ where w, z are the unknowns here. Since

$$u_x = v_w w_x + v_z z_x \quad \text{and} \quad u_y = v_w w_y + v_z z_x$$

Using this if we rewrite our PDE in terms of v, w, z we get

$$\begin{aligned} u_x + 2u_y - 4u &= v_w w_x + v_z z_x + 2(v_w w_y + v_z z_y) - 4v \\ &= v_w 2 + v_z 0 + 2(v_w(-1) + v_z 1) - v \\ &= 2v_w - 2v_w + v_z - v. \end{aligned}$$

The right hand side becomes

$$e^{x+y} = e^{\frac{1}{2}(w+z)+z} = e^{\frac{w}{2}+\frac{3z}{2}}.$$

Hence our new PDE is

$$2v_z - 4v = e^{\frac{w}{2}+\frac{3z}{2}}.$$

If we rewrite the PDE we have first $v_z - 2v = 1/2(e^{\frac{w}{2}+\frac{3z}{2}})$ and then integrating factor for left hand side is $\mu(z) = e^{\int -2dz} = e^{-2z}$. We multiply both sides with this integrating factor to get

$$e^{-2z} v_z - e^{-2z} 2v = \frac{1}{2} e^{\frac{w}{2}+\frac{3z}{2}-2z}.$$

Now the left hand side can be written as

$$(e^{-2z} v)_z = \frac{1}{2} e^{\frac{w}{2}-\frac{z}{2}}.$$

If we integrate both sides with respect to z we get

$$e^{-2z} v(w, z) = \int (e^{-2z} v)_z dz = \int \frac{1}{2} e^{\frac{w}{2}-\frac{z}{2}} dz = -e^{\frac{w}{2}-\frac{z}{2}} + f(w)$$

for some $f \in C^1$. Equivalently, $v(w, z) = -e^{\frac{w}{2}+\frac{3z}{2}} + e^{2z} f(w)$. Now if we return everything back to u, x, y we have

$$u(x, y) = v(w, z) = -e^{\frac{w}{2}+\frac{3z}{2}} + e^{2z} f(w) = -e^{\frac{2x-y}{2}+\frac{3y}{2}} + e^{2y} f(2x - y) = -e^{x+y} + e^{2y} f(2x - y)$$

where $f \in C^1$ arbitrary function.

Question 2 (Exercise 2.1, 1d) *Find the general solution of the following PDE where* $u = u(x, y)$ *and*

$$3u_x - 4u_y = x + e^x.$$

Solution: We use the idea developed in section 2.1; and do the following change of variables;

$$\begin{cases} w = (-4x - 3y = \text{this is same as}) = 4x + 3y, & x = \frac{1}{4}(w - 3z) \\ z = y, & y = z \end{cases}$$

We look for $v(w,z) = u(x,y) = u(1/4(w-3z), z)$ where w, z are the unknowns here. Since

$$u_x = v_w w_x + v_z z_x = v_w 4 + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w 3 + v_z$$

Using this if we rewrite our PDE in terms of v, w, z we get

$$3u_x - 4u_y = 3(4v_w) - 4(v_w 3 + v_z) = 12v_w - 12v_w - 4v_z = -4v_z.$$

Now we need to right $x + e^x$ (which is the right-hand side of our original PDE) in terms of w and z.

$$x + e^x = \frac{1}{4}(w - 3z) + e^{\frac{1}{4}(w-3z)}$$

Hence our new PDE is

$$-4v_z = \frac{1}{4}(w - 3z) + e^{\frac{1}{4}(w-3z)}.$$

Integrating both sides with respect to z we get

$$4v(w,z) = \int (\frac{1}{4}(w - 3z) + e^{\frac{1}{4}(w-3z)})dz = \frac{1}{4}(wz - \frac{3}{2}z^2) - \frac{4}{3}e^{\frac{1}{4}(w-3z)} + f(w)$$

for arbitrary $f \in C^1$. If we convert everything back to x, y and divide both sides by 4 we have

$$u(x,y) = v(w,z) = \frac{1}{4}(wz - \frac{3}{2}z^2) - \frac{4}{3}e^{\frac{1}{4}(w-3z)} + f(w) = \frac{1}{4}((4x+3y)y - \frac{3}{2}y^2) - \frac{4}{3}e^x + f(4x+3y).$$

for arbitrary $f \in C^1$.

Question 3 (Exercise 2.1, 2a) *Find a particular solution to the PDE in Problem 1c satisfying*

$$u(x,0) = \sin(x^2).$$

Solution: The only unknown in the solution we found in problem 1c is the function f. Using this given information we will find f. Since

$$u(x,y) = -e^{x+y} + e^{2y} f(2x - y)$$

hence

$$\sin(x^2) = u(x,0) = -e^x + f(2x)$$

From this we get $f(2x) = \sin(x^2) + e^x$. But we want to find $f(x)$ first which is easy as replace $2x = w$ (hence $x = w/2$) to get $f(x) = \sin((x/2)^2) + e^{x/2}$. Hence

$$u(x,y) = -e^{x+y} + e^{2y} f(2x - y) = -e^{x+y} + e^{2y}(\sin((x - y/2)^2) + e^{x - y/2})$$

is the particular solution we are looking for.

Question 4 (Exercise 2.1, 3) *Show that the PDE $u_x + u_y - u = 0$ with side condition $u(x,x) = \tan(x)$ has no solution.*

Solution: Suppose there is a solution (and hope to get a contradiction at the end) $u(x,y)$. Therefore, let

$$\begin{cases} w = x - y, & x = w + z \\ z = y, & y = z \end{cases}$$

We look for $v(w,z) = u(x,y) = u(w+z,z)$ where w,z are the unknowns here. If we rewrite our PDE in terms of v,w,z we get

$$u_x + u_y - u = v_w w_x + v_z z_x + v_w w_y + v_z z_y - v = v_w + v_z 0 + (v_w(-1) + v_z 1) - v = v_z - v = 0$$

Now we can find the integrating factor $\mu(z) = e^{\int(-1)dz} = e^{-z}$ and by multiplying both sides we get

$$e^{-z}v_z - e^{-z}v = 0 \quad \text{equivalently} \quad (e^{-z}v)_z = 0.$$

By integrating both sides we get

$$e^{-z}v(w,z) = \int (e^{-z}v)_z dz = f(w)$$

Hence we have $v(w,z) = e^z f(w)$. Now if we convert everything back to u,x,y we have

$$u(x,y) = v(w,z) = e^z f(w) = e^y f(x-y).$$

Using the given condition $u(x,x) = \tan(x)$ and the possible solution we found we get

$$u(x,x) = e^x f(x-x) = e^x f(0) = \tan(x)$$

from which we get $f(0) = e^{-x}\tan(x)$. Clearly, there is no f satisfying this as left hand side is constant and the right hand side is a function of x. Therefore, there is no such f satisfying this and we conclude that there is no solution to this pde with the given condition.

Question 5 (Exercise 2.1, 8) *(a) Show that the PDE $u_x = 0$, $u = u(x,y)$, has no solution which is C^1 everywhere and satisfies the side condition $u(x,x^2) = x$.*

(b) Find a solution of the problem in (a) which is valid in the first quadrant $x > 0, y > 0$.

(c) Explain the results of (a) and (b) in terms of the intersections of the side condition curve and the characteristic lines.

Solution: (a) First we shall solve the PDE $u_x = 0$. Integrating this with respect to x we get

$$u(x,y) = \int u_x dx = f(y).$$

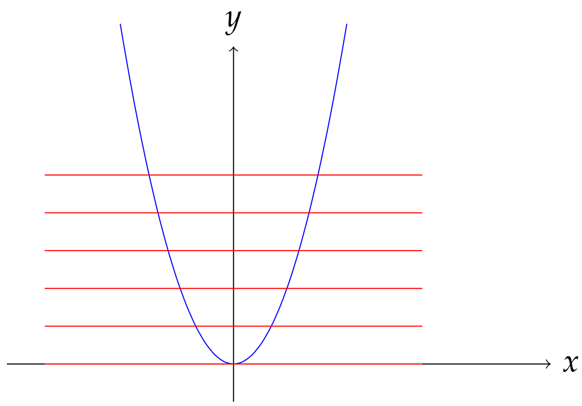
where $f \in C^1$ arbitrary function. Now we use the side condition, which we will get

$$u(x,x^2) = f(x^2) = x$$

From this we get $f(x) = \sqrt{|x|}$. Hence $u(x,y) = f(y) = \sqrt{|y|}$. However, $\sqrt{|y|}$ is not differentiable at $y = 0$. Therefore, there is no solution to above PDE with given side condition.

(b) Now if the domain we are interested in does not contain $(0,0)$ then we will have a C^1 solution. As it is asked than in the first quadrant $x > 0, y > 0$ we have $u(x,y) = \sqrt{y}$ which is C^1 everywhere in the first quadrant hence it is the solution we are looking for.

(c) Notice that the curve in the side condition has equation $y = x^2$ and the characteristic line has equation $y = b$. As we can see from the graph, they intersect more than once. They intersect at $(0,0)$ only once but at that point they do not intersect transversely, i.e., tangent to $y = b$ is parallel to the tangent of $y = x^2$ at $(0,0)$.



Question 6 (Exercise 2.2, 1a) *Obtain the general solution of the following PDE*

$$xu_x + 2yu_y = 0 \quad \textit{for } x > 0,\ y > 0.$$

Solution: In notation from section 2.2, we have $a(x,y) = x$ and $b(x,y) = 2y$. In order to make change of variables, we are looking for curves whose tangent at (x,y) is $b(x,y)/a(x,y)$ which will be parallel to $g(x,y) = a(x,y)\mathbf{i} + b(x,y)$. That is

$$\frac{dy}{dx} = \frac{b(x,y}{a(x,y)} = \frac{2y}{x}.$$

We can solve this ordinary differential equation to find the curve we are looking for. Hence, we first get

$$\frac{dy}{y} = 2\frac{dx}{x} \quad \text{equivalently} \quad \ln|y| = \ln|x|^2 + c$$

where we did integration to get this and c is arbitrary constant. Notice that our domain of interest is $x > 0, y > 0$. Hence we have $\ln y = \ln|x|^2 + c$ or

$$\ln\frac{y}{x^2} = \ln y - \ln x^2 = c \quad \text{equivalently} \quad \frac{y}{x^2} = c$$

Hence, we make the following change of variables;

$$\begin{cases} w = \frac{y}{x^2} \\ z = y \end{cases}$$

We let $v(w,z) = u(x,y)$ and we now rewrite our PDE in terms of v and its derivatives in terms of w,z. To this end, we first compute

$$u_x = v_w w_x + v_z z_x = v_w(-2\frac{y}{x^3}) + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w\frac{1}{x^2} + v_z$$

Now

$$xu_x + 2yu_y = xv_w(-2\frac{y}{x^3}) + 2y(v_w\frac{1}{x^2} + v_z) = -\frac{2y}{x^2}v_w + \frac{2y}{x^2}v_w + 2yv_z = 0.$$

As $y > 0$, from this we get $v_z = 0$ or equivalently $v(w,z) = f(w)$. Now we convert everything back to x, y, u.

$$u(x,y) = v(w,z) = f(w) = f(\frac{y}{x^2}).$$

Question 7 (Exercise 2.2, 2a) *Find the particular solution of the PDE you obtained in 1a satisfying following side condition*

$$u(x, \frac{1}{x}) = x,\ x > 0.$$

Solution: Since our general solution for the previous problem is $u(x,y) = f(\frac{y}{x^2})$, we need to find f such that u is on $(x, 1/x)$ or $y = 1/x$ has value x. to this end,

$$x = u(x, \frac{1}{x}) = f(\frac{1}{x}\frac{1}{x^2}) = f(\frac{1}{x^3})$$

From this we get $f(x) = \frac{1}{x^{1/3}}$. Hence

$$u(x,y) = f(\frac{y}{x^2}) = \frac{1}{(\frac{y}{x^2})^{1/3}} = \frac{x^{2/3}}{y^{1/3}}.$$

Question 8 (Exercise 2.2, 1d) *Obtain the general solution of the following PDE*

$$yu_x - 4xu_y = 2xy \quad \textit{for} \quad \textit{for all } (x,y).$$

Solution: In notation from section 2.2, we have $a(x,y) = y$ and $b(x,y) = -4x$. In order to make change of variables, we are looking for curves whose tangent at (x,y) is $b(x,y)/a(x,y)$ which will be parallel to $g(x,y) = a(x,y)\mathbf{i} + b(x,y)$. That is

$$\frac{dy}{dx} = \frac{b(x,y}{a(x,y)} = \frac{-4x}{y}.$$

We can solve this ordinary differential equation to find the curve we are looking for. Hence, we first get

$$ydy = -4xdx \quad \text{equivalently} \quad \frac{y^2}{2} = -2x^2 + c$$

That is, $y^2 + 4x^2 = c$. Hence, we make the following change of variables;

$$\begin{cases} w = y^2 + 4x^2 \\ z = y \end{cases}$$

We let $v(w,z) = u(x,y)$ and we now rewrite our PDE in terms of v and its derivatives in terms of w, z. To this end, we first compute

$$u_x = v_w w_x + v_z z_x = v_w 8x + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w 2y + v_z$$

Now we rewrite our PDE in terms of v and its derivatives with respect to w, z

$$2xy = yu_x - 4xu_y = y(8xv_w) - 4x(v_w 2y + v_z) = 8xyv_w - 8xyv_w - 4xv_z.$$

We have $2xy = -4xv_z$. From this we get $v_z = -2y = -2z$ whenever $x \neq 0$. Integrating with respect to z in $v_z = -2z$ we get

$$v(w,z) = -z^2 + f(w) \quad \text{for arbitrary } f \in C^1.$$

Hence we get our solution $u(x,y)$ by converting everything back to u, x, y

$$u(x,y) = v(x,y) = -z^2 + f(w) = -y^2 + f(y^2 + 4x^2) \quad \text{for arbitrary } f \in C^1.$$

Question 9 (Exercise 2.2, 2d) *Find the particular solution of the PDE in 1d with side condition*

$$u(x,0) = x^4.$$

Solution: As we get $u(x,y) = -y^2 + f(y^2 + 4x^2)$ we then use this and given side condition to find f;

$$x^4 = u(x,0) = 0 + f(0 + 4x^2).$$

From this we get $f(x) = x^2/16$. Hence

$$u(x,y) = -y^2 + f(y^2 + 4x^2) = -y^2 + \frac{1}{16}(y^2 + 4x^2)^2.$$