UCONN - Math 3435 - Spring 2018 - Problem set 4

Question 1 (Exercise 3.1, 3a) *Solve the following Heat conduction problem*

$$\begin{cases} u_t = 2u_{xx}, \quad 0 \leq x \leq 3, \quad t \geq 0 \\ u(0,t) = 0 \quad \textit{and} \quad u(3,t) = 0 \\ u(x,0) = 5\sin(4\pi x) + 2\sin(10\pi x). \end{cases}$$

Solution:

- By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as c).

Solution: Rewrite the PDE as $2u_{xx} = u_t$. Let $u(x,t) = X(x)T(t)$. Then

$$u_{xx} = X''T \quad \text{and} \quad u_t = XT'.$$

Substitute this in to the differential equation to get

$$2u_{xx} - u_t = 2X''T - XT' = 0 \quad \text{equivalently} \quad \frac{X''}{X} = \frac{T'}{2T} = c.$$

Hence

$$\frac{X''}{X} = c \quad \rightarrow \quad X'' - cX = 0,$$
$$\frac{T'}{2T} = c \quad \rightarrow \quad T' - 2cT = 0.$$

- Rewrite the boundary values in terms of X and T.

Solution: We have at $x = 0$ as $u(x,t) = X(x)T(t)$ then

$$u(0,t) = X(0)T(t) = 0; \quad \text{one has either} \quad X(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = 3$

$$u(3,t) = X(3)T(t) = 0; \quad \text{one has either} \quad X(3) = 0 \quad \text{or} \quad T(t) = 0.$$

- Now choose the boundary values which will give a non-trivial solution and write the ordinary differential equation corresponding to X.

Solution: We know that the choice of $T(t) = 0$ gives only the trivial solution as $u(x,t) = X(x)T(t) = 0$.
Therefore, we choose our boundary conditions as $X(0) = 0$ and $X(3) = 0$ in order to obtain the non-trivial solution. Now if we rewrite the ordinary differential equation corresponding to X we get

$$X'' - cX = 0, \quad X(0) = 0 \quad \text{and} \quad X(3) = 0.$$

- Solve the two-point boundary value problem corresponding to X. Find all eigenvalues c and corresponding eigenfunctions X.

For $c = 0$, the ordinary differential equation $X'' - cX = 0$ becomes $X'' = 0$. Then the solution is

$$X(x) = ax + b \quad \text{for some constants} \quad a, b.$$

Using the boundary conditions $X(0) = 0$ we first get $X(0) = b = 0$ and we have $X(x) = ax$. Next using $X(3) = 0$ we get $a = 0$. Hence we have $X(x) = 0$ which leads us to trivial solution as $u(x,t) = X(x)T(t) = 0$ in this case. Hence we do not need to solve the ODE corresponds to T.

For $c > 0$. Say $c = \lambda^2$ for some $\lambda > 0$. Then

$$X'' - cX = X'' - \lambda^2 X = 0.$$

This differential equation has characteristic equations $r^2 - \lambda^2 = 0$, hence roots are $r = \pm\lambda$. This gives the solution

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}.$$

Using boundary conditions $X(0) = 0$ first we get

$$X(0) = Ae^0 + Be^0 = 0 \quad \text{which gives us} \quad B = -A.$$

On the other hand, $X(3) = 0$ we get

$$0 = X(3) = Ae^{3\lambda} + Be^{-3\lambda} = Ae^{3\lambda} - Ae^{-3\lambda}$$

Which gives us $Ae^{3\lambda} = Ae^{-3\lambda}$ which is possible only if $A = 0$. Hence we get $A = -B = 0$. This leads us $X(x) = 0$ the trivial solution again. which leads us to $X(x) = 0$. Hence we get the trivial solution. Hence we do not need to solve the ODE corresponds to T.

For $c < 0$. Let $c = -\lambda^2 < 0$ for some $\lambda > 0$, we get

$$X'' - cX = X'' + \lambda^2 X = 0.$$

We know that the solution is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

Using the boundary condition $X(0) = 0$, we get

$$0 = X(0) = A\cos(0 + B\sin(0) = A\cos(0) = A$$

We get $A = 0$. We have now $X(x) = B\sin(\lambda x)$. Using $X(3) = 0$ we get

$$0 = X(3) = B\sin(\lambda 3)$$

We know that sine function vanishes at points integer multiple of π i.e. $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots,$. Hence we should have $3\lambda = n\pi$. From this we get $\lambda = n\pi/3$. These values of $\lambda = \lambda_n = n\pi/3$ known as eigenvalues. For these values of λ, $X(x) = \sin(\lambda x) = \sin(n\pi/3x)$ solve above ODE corresponds to X. These functions are known as eigenfunctions. Hence, we will put dependence on n as we have $n = 1, 2, \ldots,$ as eigenvalues and corresponding eigenfunctions

$$\lambda = \frac{n\pi}{3} \quad \text{and} \quad X_n(x) = B_n \sin(\lambda x) = B_n \sin(\frac{n\pi}{3}x).$$

For this values of $c = -\lambda^2 = -(\frac{n\pi}{3})^2$ we will solve the ODE corresponds to T.

- For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

Solution: We have no solution for $c = 0$ and $c > 0$, (those solutions are only $X(x) = 0$ which is trivial solutions). We only need to take care when $c = -\lambda^2 = -(\frac{n\pi}{3})^2 < 0$ for $n = 1, 2, \ldots$. For this value of c the ordinary differential equation corresponding to T is now

$$T' + 2cT = T' + 2(\frac{n\pi}{3})^2 T = 0.$$

Since for each n we have a different solution T we shall show this dependence by subscript n, T_n.

$$T'_n + 2\frac{n^2\pi^2}{3^2}T_n = 0.$$

We know that this is a first order linear ordinary differential equation and its solution is

$$T_n(t) = C_n e^{-2\frac{n^2\pi^2}{3^2}t}.$$

for some C_n.

- Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

Solution: We know that the solution for each $n = 1, 2, \ldots$ is

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin(\frac{n\pi}{3}x)C_n e^{-2\frac{n^2\pi^2}{3^2}t} = D_n \sin(\frac{n\pi}{3}x)e^{-2\frac{n^2\pi^2}{3^2}t}.$$

where we rename constant $B_nC_n = D_n$ The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x)e^{-2\frac{n^2\pi^2}{3^2}t}.$$

- Using the given initial value and the general solution you found in (f), find the particular solution.

Now the initial value is given as $u(x,0) = 5\sin(4\pi x) + 2\sin(10\pi x)$. Hence, plug in $t = 0$ in the solution we have in (f) gives us

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x)e^0 = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x) = 5\sin(4\pi x) + 2\sin(10\pi x).$$

From this we get that all $D_n = 0$ except $D_{12} = 5$ and $D_{30} = 2$ as when $n = 12$ above we get $D_{12}\sin(\frac{12\pi}{3}x) = D_{12}\sin(4\pi x)$ which should be $5\sin(4\pi x)$. Similarly, when $n = 30$ we have $D_{30}\sin(\frac{30\pi}{3}x) = D_{30}\sin(30\pi x)$ which should be $2\sin(10\pi x)$. Hence for $n = 12$ and $n = 30$ we get

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x)e^{-2\frac{n^2\pi^2}{3^2}t} \\ &= 5\sin(\frac{12\pi}{3}x)e^{-2\frac{12^2\pi^2}{3^2}t} + 2\sin(\frac{30\pi}{3}x)e^{-2\frac{30^2\pi^2}{3^2}t} \end{aligned}$$

which is the solution we are looking for.

Question 2 (Exercise 3.1, 3d) *Solve the following Heat conduction problem*

$$\begin{cases} u_t = 2u_{xx}, & 0 \le x \le 3, \quad t \ge 0 \\ u(0,t) = 0 & \text{and} \quad u(3,t) = 0 \\ u(x,0) = -9\cos(\frac{\pi}{6}(2x+3)). \end{cases}$$

Solution: This problem is exactly the same problem except the initial condition. Hence our steps are up to the last line exactly the same. The only difference will be finding D_n. So the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x)e^{-2\frac{n^2\pi^2}{3^2}t}.$$

We will find D_n using the initial condition $u(x,0) = -9\cos(\frac{\pi}{6}(2x+3))$. If we look carefully, this is a function of cosine whereas our solution contains only sine. Remember that

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Using this above with $a = 2x/6 = x/3$ and $b = 3\pi/6 = \pi/2$ we have

$$\cos(\frac{\pi}{6}(2x+3)) = \cos((2x\frac{\pi}{6} + 3\frac{\pi}{6})) = \cos((\frac{x\pi}{3} + \frac{\pi}{2})) = \cos(\frac{x\pi}{3})\cos(\frac{\pi}{2}) - \sin(\frac{x\pi}{3})\sin(\frac{\pi}{2})$$

Remember that $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$ we get

$$\cos(\frac{\pi}{6}(2x+3)) = -\sin(\frac{x\pi}{3}).$$

Hence we have the initial condition

$$u(x,0) = -9\cos(\frac{\pi}{6}(2x+3)) = 9\sin(\frac{x\pi}{3}).$$

We use this and the general solution we got above to figure out constants D_n.

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x)e^0 = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x) = 9\sin(\frac{x\pi}{3})$$

From this we see that $D_1 = 9$ and all other $D_n = 0$. Hence

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi}{3}x)e^{-2\frac{n^2\pi^2}{3^2}t} = 9\sin(\frac{x\pi}{3})e^{-2\frac{1^2\pi^2}{3^2}t}$$

is the solution we are looking for.

Question 3 (Exercise 3.1, 6b) *Solve the following Heat conduction problem*

$$\begin{cases} u_t = u_{xx}, \quad -\pi \le x \le \pi, \quad t \ge 0 \\ u(-\pi,t) = u(\pi,t) \quad \textit{and} \quad u_x(-\pi,t) = u_(\pi,t) \\ u(x,0) = \frac{1}{2} + \cos(2x) - 6\sin(2x). \end{cases}$$

Solution: (See solution to the next problem). Since the next problem and this problem are the same except the initial condition we have exactly the same steps up to the general solution which is $U(x,t) =$ solution corresponds to $c = 0 +$ solution corresponds to $c < 0$

$$u(x,t) = bd + \sum_{n=1}^{\infty} u_n(x,t) = bd + \sum_{n=1}^{\infty} [A_n\cos(nx) + B_n\sin(nx)]e^{-n^2t}.$$

We now use the given initial conditions to find A_n and B_n and bd. At $t = 0$ we have $u(x,0) = \frac{1}{2} + \cos(2x) - 6\sin(2x)$ and using above solution we get

$$u(x,0) = bd + \sum_{n=1}^{\infty}[A_n\cos(nx) + B_n\sin(nx)]e^0 = bd + \sum_{n=1}^{\infty}[A_n\cos(nx) + B_n\sin(nx)]e^0 = \frac{1}{2} + \cos(2x) - 6\sin(2x)$$

Notice that the only constant on the right hand side is $1/2$ and the only constant on the left hand side is bd. Hence we should have $bd = 1/2$. The coefficient of $\cos(2x)$ on the right hand side is 1 and it is A_2. Hence we should have $A_2 = 1$. Similarly, the coefficient of $\sin(2x)$ is -6 on the right hand side and it is B_2 on the left hand side. Hence we should have $B_6 = -6$. All other A_n and B_n are zero. We have now

$$\begin{aligned} u(x,t) &= bd + \sum_{n=1}^{\infty}[A_n\cos(nx) + B_n\sin(nx)]e^{-n^2t} \\ &= \frac{1}{2} + e^{-2^2t}\cos(2x) - 6e^{-2^2t}\sin(2x) \end{aligned}$$

is the solution we are looking for.

Question 4 (Exercise 3.1, 6d)

$$\begin{cases} u_t = u_{xx}, \quad -\pi \le x \le \pi, \quad t \ge 0 \\ u(-\pi,t) = u(\pi,t) \quad \textit{and} \quad u_x(-\pi,t) = u_(\pi,t) \\ u(x,0) = 6\sin(x) - 7\cos(3x) - 7\sin(3x). \end{cases}$$

Solution:

- By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as c).

Solution: Rewrite the PDE as $u_{xx} = u_t$. Let $u(x,t) = X(x)T(t)$. Then

$$u_{xx} = X''T \quad \text{and} \quad u_t = XT'.$$

Substitute this in to the differential equation to get

$$u_{xx} - u_t = X''T - XT' = 0 \quad \text{equivalently} \quad \frac{X''}{X} = \frac{T'}{T} = c.$$

Hence

$$\frac{X''}{X} = c \quad \to \quad X'' - cX = 0,$$
$$\frac{T'}{T} = c \quad \to \quad T' - cT = 0.$$

- Rewrite the boundary values in terms of X and T and choose the boundary values which will give a non-trivial solution and write the ordinary differential equation corresponding to X and T.

Solution: We have at $x = \pi$ and $x = \pi$ as $u(x,t) = X(x)T(t)$ then

$$u(-\pi,t) = X(-\pi)T(t) \quad \text{and} \quad u(\pi,t) = X(\pi)T(t)$$

Since $u(-\pi,t) = u(\pi,t)$ we have $X(-\pi)T(t) = X(\pi)T(t)$. From this we see that if $T(t) = 0$ then we have the trivial solution. Otherwise, we have $X(-\pi) = X(\pi)$. Similarly

$$u_x(-\pi,t) = X'(-\pi)T(t) \quad \text{and} \quad u_x(\pi,t) = X'(\pi)T(t)$$

Since $u_x(-\pi,t) = u_x(\pi,t)$ we have $X'(-\pi)T(t) = X'(\pi)T(t)$. Similarly as above, from this we see that if $T(t) = 0$ then we have the trivial solution. Otherwise, we have $X'(-\pi) = X'(\pi)$. Hence we have

$$\frac{X''}{X} = c \quad \to \quad X'' - cX = 0, \quad \text{and} X(-\pi) = X(\pi), \quad X'(-\pi) = X'(\pi),$$
$$\frac{T'}{T} = c \quad \to \quad T' - cT = 0.$$

- Solve the two-point boundary value problem corresponding to X. Find all eigenvalues c and corresponding eigenfunctions X.

For $c = 0$, the ordinary differential equation $X'' - cX = 0$ becomes $X'' = 0$. Then the solution is

$$X(x) = ax + b \quad \text{for some constants} \quad a, b.$$

Since $X(-\pi) = a(-\pi) + b = X(\pi) = a\pi + b$. This gives us $a = 0$. So we have $X(x) = b$. The second condition, $X'(-\pi) = X'(\pi)$ is automatically satisfies as $X'(x) = 0 = X'(-\pi) = X'(\pi)$. Hence when

$c = 0$ we have $X(x) = b$. For this value of $c = 0$ we solve ODE $T' - cT = 0$ corresponds to T. That is $T' = 0$ which has solution $T(t) = d$.

For $c > 0$. Say $c = \lambda^2$ for some $\lambda > 0$. Then

$$X'' - cX = X'' - \lambda^2 X = 0.$$

This differential equation has characteristic equations $r^2 - \lambda^2 = 0$, hence roots are $r = \pm\lambda$. This gives the solution

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}.$$

Using boundary conditions $X(-\pi) = X(\pi)$ first we get

$$Ae^{-\lambda\pi} + Be^{\lambda\pi} = Ae^{\lambda\pi} + Be^{-\lambda\pi}$$

Doing a little algebra we get

$$(A - B)e^{-\lambda\pi} = (A - B)e^{\lambda\pi}$$

This can happen only if $A - B = 0$ or $A = B$. Now if we use $X'(-\pi) = X'(\pi)$ we get

$$A\lambda e^{-\lambda\pi} - B\lambda e^{\lambda\pi} = A\lambda e^{\lambda\pi} - \lambda Be^{-\lambda\pi}$$

Doing again a little algebra we get (remember $\lambda > 0$ so we can cancel them in both sides)

$$(A + B)e^{-\lambda\pi} = (A + B)e^{\lambda\pi}$$

Again this can happen only of $A + B = 0$ or $A = -B$. Now combining this with $A = B$ we get $A = B = 0$. This gives us in turn $X(x) = 0$ which is trivial solution. Hence we do not need to solve ODE corresponds to T.

For $c < 0$. Let $c = -\lambda^2 < 0$ for some $\lambda > 0$, we get

$$X'' - cX = X'' + \lambda^2 X = 0.$$

We know that the solution is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

Now we use $X(-\pi) = X(\pi)$ first;

$$A\cos(-\lambda\pi) + B\sin(-\lambda\pi) = A\cos(\lambda\pi) + B\sin(\lambda\pi)$$

Using $\cos(x) = \cos(-x)$ and $-\sin(x) = \sin(-x)$ we get

$$A\cos(\lambda\pi) - B\sin(\lambda\pi) = A\cos(\lambda\pi) + B\sin(\lambda\pi)$$

and thereupon we conclude that (after simplifying cosine terms) $2B\sin(\lambda\pi) = 0$. Now either $B = 0$ or $\sin(\lambda\pi) = 0$. The latter can happen when $\lambda\pi = n\pi$ whenever $n = 1, 2, \ldots$. Hence we get $\lambda = n$.

Now using $X'(-\pi) = X'(\pi)$ we get

$$-A\lambda\sin(-\lambda\pi) + B\lambda\cos(-\lambda\pi) = -A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi)$$

Using $\cos(x) = \cos(-x)$ and $-\sin(x) = \sin(-x)$ again we get

$$A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi) = -A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi)$$

From this we get (this time canceling cosine terms on both sides) we get $2A\lambda\sin(\lambda\pi) = 0$ which can happen either $A = 0$ or $\sin(\lambda\pi) = 0$. Once again the latter can happen when $\lambda\pi = n\pi$ whenever $n = 1, 2, \ldots$. Hence we get $\lambda = n$. Hence the solution for every fixed $\lambda = n$ is

$$X(x) = A\cos(nx) + B\sin(nx)$$

and let us use X_n to show that we have solution for every $n = 1, 2, \dots$ (likewise for the constants A_n and B_n) and write it as

$$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \quad \text{for} \quad n = 1, 2, \dots.$$

Note that we did not consider what happens either $A = 0$ or $B = 0$ as we consider more general solution which contains those cases.

For this values of $c = -\lambda^2 = -n^2$ we will solve the ODE corresponds to T.

- For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

Solution: We have already solved the ODE corresponds to T when $c > 0$. Remember we have trivial solution when $c > 0$ (those solutions are only $X(x) = 0$ which is trivial solutions). We only need to take care when $c = -\lambda^2 = -n^2 < 0$ for $n = 1, 2, \dots$. For this value of c the ordinary differential equation corresponding to T is now

$$T' + cT = T' + 2n^2 T = 0.$$

Since for each n we have a different solution T we shall show this dependence by subscript n, T_n.

$$T_n' + n^2 T_n = 0.$$

We know that this is a first order linear ordinary differential equation and its solution is

$$T_n(t) = C_n e^{-n^2 t} \quad n = 1, 2, \dots$$

for some C_n.

- Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

Solution: We know that the solution for each $n = 1, 2, \dots$ corresponds to $c = -\lambda^2 = -n^2$ is

$$u_n(x,t) = X_n(x)T_n(t) = [A_n \cos(nx) + B_n \sin(nx)]C_n e^{-n^2 t} = [A_n \cos(nx) + B_n \sin(nx)]e^{-n^2 t}.$$

where we just hide constant B_n in A_n and B_n.

On the other hand when $c = 0$ we have $X(x) = b$ and $T(t) = d$, hence in this case we have $u_0(x,t) = X(x)T(t) = bd$ (let separate by using subscript 0 with the one for $c < 0$).

The general solution is $U(x,t) =$ solution corresponds to $c = 0 +$ solution corresponds to $c < 0$

$$u(x,t) = bd + \sum_{n=1}^{\infty} u_n(x,t) = bd + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]e^{-n^2 t}.$$

- Using the given initial value and the general solution you found in (f), find the particular solution.

Now the initial value is given as $u(x,0) = 6\sin(x) - 7\cos(3x) - 7\sin(3x)$. Hence, plug in $t = 0$ in the solution we have in (f) gives us

$$\begin{aligned} u(x,0) &= bd + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]e^0 = bd + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)] \\ &= 6\sin(x) - 7\cos(3x) - 7\sin(3x). \end{aligned}$$

The first thing we should realize is that on the left hand side the only constant term is bd and there is no constant term on the right hand side. This tells us $bd = 0$. Coefficient of $\sin(x)$ on the right hand side is 6 and it is B_1 on the left. Hence $B_1 = 6$. Similarly, the coefficient of $\cos(3x)$ on the right is -7 and it is A_3 on the left hand side. Hence $A_3 = -7$. Finally, coefficient of $\sin(3x)$ is -7 on the right hand side and it is B_3 on the right hand side. Hence $B_3 = 07$. All other $A_n = 0$ and all other $B_n = 0$. Using this in the solution we found above, we get

$$\begin{aligned} u(x,t) &= bd + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)] e^{-n^2 t} \\ &= 6\sin(x)e^{-1^2 t} - 7\cos(3x)e^{-3^2 t} - 7\sin(3x)e^{-3^2 t} \end{aligned}$$

which is the solution we are looking for.

Question 5 *Solve the following Heat conduction problem*

$$\begin{cases} 9u_{xx} = u_t, \quad 0 < x < 3, \quad t > 0 \\ u_x(0,t) = 0 \quad \textit{and} \quad u_x(3,t) = 0 \\ u(x,0) = 2\cos(\frac{\pi x}{3}) - 4\cos(\frac{5\pi x}{3}). \end{cases}$$

Solution:

- By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

Solution: Rewrite the PDE as $9u_{xx} - u_t = 0$. Let $u(x,t) = X(x)T(t)$. Then

$$u_{xx} = X''T \quad \text{and} \quad u_t = XT'.$$

Substitute this in to the differential equation to get

$$9u_{xx} - u_t = 9X''T - XT' = 0 \quad \text{equivalently} \quad \frac{X''}{X} = \frac{T'}{9T} = -\lambda.$$

Hence

$$\begin{aligned} \frac{X''}{X} &= -\lambda \quad \rightarrow \quad X'' + \lambda X = 0, \\ \frac{T'}{9T} &= -\lambda \quad \rightarrow \quad T' + 9\lambda T = 0. \end{aligned}$$

- Rewrite the boundary values in terms of X and T. **Be careful on the boundary conditions for u_x.**

Solution: We have at $x = 0$ as $u_x(x,t) = X'(x)T(t)$ then

$$u_x(0,t) = X'(0)T(t) = 0; \quad \text{one has either} \quad X'(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = \pi$

$$u_x(3,t) = X'(\pi)T(t) = 0; \quad \text{one has either} \quad X'(3) = 0 \quad \text{or} \quad T(t) = 0.$$

- Now choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.

Solution: We know that the choice of $T(t) = 0$ gives only the trivial solution as $u(x,t) = X(x)T(t) = 0$.

Therefore, we choose our boundary conditions as $X'(0) = 0$ and $X'(3) = 0$ in order to obtain the non-trivial solution. Now if we rewrite the ordinary differential equation corresponding to X we get

$$X'' + \lambda X = 0, \quad X'(0) = 0 \quad \text{and} \quad X'(3) = 0.$$

- Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.

For $\lambda = 0$, the ordinary differential equation $X'' + \lambda X = 0$ becomes $X'' = 0$. Then the solution is

$$X(x) = ax + b \quad \text{for some constants} \quad a, b.$$

Using the boundary conditions $X'(0) = 0 \quad \text{and} \quad X'(3) = 0$ we get $X'(x) = a$ and then $X'(0) = a = 0$. Hence we have $a = 0$. The second boundary conditions is also verified as $X'(3) = a = 0$. Hence $X(x) = b$ is solution corresponding to the eigenvalue $\lambda = 0$.

For $\lambda < 0$. Say $\lambda = -n^2$ for some constant $n > 0$. Then

$$X'' + \lambda X = X'' - n^2 X = 0.$$

This differential equation has characteristic equations $r^2 - n^2 = 0$, hence roots are $r = \pm n$. This gives the solution

$$X(x) = Ae^{nx} + Be^{-nx}.$$

Take derivative to get

$$X'(x) = Ane^{nx} - Bne^{-nx}.$$

Using boundary conditions we get

$$X'(0) = Ane^0 - Bne^0 = 0 \quad \text{and} \quad X'(3) = Ane^{3n} - Bne^{3n} = 0$$

which leads us to $X(x) = 0$. Hence we get the trivial solution.

For $\lambda > 0$. Let $\lambda = k^2$ for some constant $k > 0$, we get

$$X'' + \lambda X = X'' + k^2 X = 0.$$

We know that the solution is

$$X(x) = A\cos(kx) + B\sin(kx).$$

Using this general solution and the first boundary condition and $X'(x) = -Ak\sin(kx) + Bk\cos(kx)$ that

$$X'(0) = -Ak\sin(0) + Bk\cos(0) = Bk = 0 \quad \text{therefore we have} \quad B = 0.$$

Using the second boundary condition, we get (as $B = 0$, $X'(x) = -Ak\sin(kx)$)

$$X'(\pi) = -Ak\sin(3k) = 0.$$

This holds when $3k = n\pi$ for $n = 1, 2, \ldots$. Hence we get $k = n\pi/3$, or equivalently, we get the eigenvalues

$$\lambda_n = k^2 = \frac{n^2\pi^2}{3^2}, \quad n = 1, 2, \ldots.$$

The corresponding eigenfunction (corresponding to λ_n) is

$$X_n(x) = \cos(\frac{n\pi x}{3}).$$

- For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

Solution: We have for $\lambda = 0$, $X_0(x) = b$ as a nontrivial solution. Hence we need to find corresponding solution T_0 for $\lambda = 0$. When $\lambda = 0$, the corresponding differential equation is

$$T' + 9\lambda T = T' = 0.$$

$T' = 0$ has solution $T(t) = b$ for some constant b. Hence for $\lambda = 0$ we have corresponding solution $T_0(t) = b$.

For $\lambda = \frac{n^2\pi^2}{3^2} > 0$ the ordinary differential equation corresponding to T is now

$$T' + 9\lambda T = 0.$$

Plug in $\lambda = \frac{n^2\pi^2}{3^2}$ we get (for each n we have a different solution T_n)

$$T_n' + 9\frac{n^2\pi^2}{3^2}T_n = T_n' + n^2\pi^2 T_n = 0.$$

We know that this is a first order linear ordinary differential equation and its solution is

$$T_n(t) = C_n e^{-n^2\pi^2 t}.$$

for some C_n.

- Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

Solution: We know that the solution for each $n > 0$ is

$$u_n(x,t) = X_n(x)T_n(t) = \cos(\frac{n\pi x}{3})C_n e^{-n^2\pi^2 t}.$$

and when $n = 0$ we have $u_0(x,t) = ab = C_0$ for some constant C_0. The general solution is

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} u_n(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3})e^{-n^2\pi^2 t}.$$

- Using the given initial value and the general solution you found in (f), find the particular solution.

Now the initial value is given as $u(x,0) = 2\cos(\frac{\pi x}{3}) - 2\cos(\frac{5\pi x}{3})$. Hence, plug in $t = 0$ in the solution we have in (f) gives us

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3})e^0 = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3}) = 2\cos(\frac{\pi x}{3}) - 2\cos(\frac{5\pi x}{3}).$$

From this we get that, $C_0 = 0$, and all $C_n = 0$ except $C_1 = 2$ and $C_5 = -2$. Hence in the solution, the only terms we have, for $n = 1, n = 5$ with $C_1 = 2$, $C_5 = -2$;

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3})e^{-n^2\pi^2 t} \\ &= 2\cos(\frac{\pi x}{3})e^{-1^2\pi^2 t} - 2\cos(\frac{5\pi x}{3})e^{-5^2\pi^2 t} \\ &= 2\cos(\frac{\pi x}{3})e^{-\pi^2 t} - 2\cos(\frac{5\pi x}{3})e^{-25\pi^2 t}. \end{aligned}$$

Question 6 (Problem 9, Page 137) *(a) You need to interpret that u satisfies the following PDE* $u_t = ku_{xx} - hu$ *as insulation at a rate proportional to the temperature of the slab we expect this.*

(b) asks you if $w(x,t)$ *satisfies the Heat equation* $w_t = kw_{xx}$ *then* $u(x,t) = e^{-ht}w(x,t)$ *satisfies that* $u_t = ku_{xx} - hu$*. Now as u is given we need to show that it satisfies by finding needed derivatives* $u_t = ku_{xx} - hu$ *or equivalently* $u_t - ku_{xx} + hu = 0$*. Since*

$$u_t(x,t) = \frac{\partial}{\partial t}(e^{-ht}w(x,t)) = -he^{-ht}w(x,t) + e^{-ht}w_t(x,t)$$
$$u_{xx}(x,t) = e^{-ht}w_{xx}(x,t)$$

Then

$$u_t - ku_{xx} + hu = -he^{-ht}w(x,t) + e^{-ht}w_t(x,t) - ke^{-ht}w_{xx}(x,t) + he^{-ht}w(x,t) = e^{-ht}w_t(x,t) - ke^{-ht}w_{xx}(x,t).$$

Since $w_t = kw_{xx}$ *we use this above to get*

$$u_t - ku_{xx} + hu = e^{-ht}w_t(x,t) - ke^{-ht}w_{xx}(x,t) = e^{-ht}kw_{xx}(x,t) - ke^{-ht}w_{xx}(x,t) = 0.$$

Hence $u(x,t) = e^{-ht}w(x,t)$ *solves* $u_t = ku_{xx} - hu$ *as long as w solves* $w_t = kw_{xx}$.

(c) Using part (b) we will find a solution to

$$\begin{cases} u_t = ku_{xx} - hu, \quad 0 < x < L, \quad t > 0 \\ u(0,t) = 0 \quad and \quad u(L,t) = 0 \\ u(x,0) = \sum_{n=1}^{N} b_n \sin(\frac{n\pi x}{L}. \end{cases}$$

Let $w(x,t)$ *be a solution to Heat equation* $w_t = kw_{xx}$*. Then we know that* $u(x,t) = e^{-ht}w(x,t)$ *solves above PDE or* $w(x,t) = e^{ht}u(x,t)$*. Now we have*

$$\begin{cases} w_t = kw_{xx}, \quad 0 < x < L, \quad t > 0 \\ w(0,t) = e^{ht}u(0,t) = 0 \quad and \quad w(L,t) = e^{ht}u(L,t) = 0 \\ w(x,0) = e^{h0}u(x,0) = \sum_{n=1}^{N} b_n \sin(\frac{n\pi x}{L}). \end{cases}$$

We know that general solution (as boundary conditions are homogeneous) to above heat equation is

$$w(x,t) = \sum_{n=1}^{\infty} C_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L}).$$

Now we use the initial condition to figure out C_n.

$$w(x,0) = \sum_{n=1}^{\infty} C_n e^0 \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{N} b_n \sin(\frac{n\pi x}{L}).$$

From this we get $C_1 = b_1$, $C_2 = b_2$, $\ldots$, $C_N = b_N$ *and all other* $C_N = 0$*. Hence we get*

$$w(x,t) = \sum_{n=1}^{N} b_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L}).$$

Since $u(x,t)$ *is the solution we are looking for and*

$$u(x,t) = e^{-ht}w(x,t) = e^{-ht} \sum_{n=1}^{N} b_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L}).$$