UCONN - Math 3435 - Spring 2018 - Problem set 5

Question 1 (Exercise 3.3, 3) *Find all product solutions of the heat equation*

$$\begin{cases} u_t = ku_{xx} & 0 \leq x \leq L,\ t \geq 0 \\ u(0,t) = 0 & u_x(L,t) = 0 \\ u(x,0) = f(x). \end{cases}$$

Solution: As the question ask, we will find all solutions of the form $u(x,t) = X(x)T(t)$. We have

$$u_t = X(x)T'(t) \quad \text{and} \quad U_{xx} = X''(x)T(t).$$

Using this in the Heat equation we get

$$X(x)T'(t) = kX''(x)T(t) \quad \text{equivalently} \quad \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

which can happen both ratios are constant. Say

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = c.$$

Then we obtain

$$T'(t) - kcT(t) = 0 \quad \text{and} \quad X''(x) - cX(x) = 0.$$

We will also rewrite the boundary conditions in terms of X, T. Since $u(0,t) = X(0)T(t) = 0$ we get $X(0) = 0$ (otherwise, if we choose $T(t) = 0$ we get zero solution). Similarly, $u_x(L,t) = X'(L)T(t) = 0$ which gives us $X'(L) = 0$. Hence we have $X(0) = 0 = X'(L)$.

We focus on $X''(x) - cX(x) = 0$ with the boundary conditions $X(0) = 0 = X'(L)$.

When $c = 0$ we have $X''(x) = 0$ which gives us $X(x) = ax + b$ for some constant a, b. Using the boundary conditions $X(0) = 0 = X'(L)$ we get $b = 0 = a$. Hence no non-trivial solution is coming from $c = 0$.

When $\lambda^2 = c > 0$, for some $\lambda > 0$, we have $X(x) = ae^{\lambda x} + be^{-\lambda x}$ for some constant a, b. Using $X(0) = 0$ we get $a = -b$ and using $X'(L) = 0$ we get

$$X'(L) = a\lambda e^{\lambda L} + a\lambda e^{-\lambda L} = 0$$

implies first $a = 0$ second $a = -b = 0$. Hence we also have trivial solution in this case.

When $-\lambda^2 = c < 0$ for some $\lambda > 0$ we then have $X(x) = a\cos(\lambda x) + b\sin(\lambda x)$ for some constant a, b. Since $X(0) = 0$ we get $a = 0$. Similarly, $X'(L) = -b\lambda\cos(\lambda L) = 0$, this can happen if $\lambda L =$ is multiple of $\pi/2 = (n + 1/2)\pi$. Hence $\lambda L = (n + 1/2)\pi$, hence $\lambda = (n + 1/2)\pi/L$ for $n = 0, 1, 2, \ldots$ which is our eigenvalue. Hence

$$\lambda_n = \frac{(n + \frac{1}{2})\pi}{L} \quad \text{and correcponding eigenfunction} \quad X_n(x) = A_n \sin(\frac{(n + \frac{1}{2})\pi x}{L}) \quad n = 0, 1, 2, \ldots$$

for some constant A_n. For this value of $c = -\lambda^2 = -((n + 1/2)\pi/L)^2$ we solve $0 = T'(t) - kcT(t) = T'(t) + k((n + 1/2)\pi/L)^2 T(t)$ which has solution

$$T_n(t) = B_n e^{-k((n + \frac{1}{2})\pi/L)^2 t} \quad n = 0, 1, 2, \ldots$$

for some constant B_n. Hence our solution is

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n B_n \sin(\frac{(n + \frac{1}{2})\pi x}{L}) e^{-k((n + \frac{1}{2})\pi/L)^2 t}$$

For simplicity call $A_nB_n = C_n$. Now we shall use the initial condition to find C_n.

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} C_n \sin(\frac{(n+\frac{1}{2})\pi x}{L}) = C_0 \sin(\frac{\pi x}{2L}) + C_1 \sin(\frac{3\pi x}{2L}) + C_2 \sin(\frac{5\pi x}{2L}) + \dots$$

where $C_0, C_1, C_2, \dots$ are constant to be found. This will give us the solution.

Question 2 (Exercise 3.3, 4) *Find all product solutions of the heat equation*

$$\begin{cases} u_t = 2u_{xx} & 0 \le x \le 1,\ t \ge 0 \\ u(0,t) = -1 & u_x(1,t) = 1 \\ u(x,0) = x + \sin(\frac{3\pi x}{2}) - 1. \end{cases} \tag{1}$$

Solution: We first should make the non-homogeneous boundary conditions homogeneous. To this end, we look for time-independent or steady-state solution $u_p(x,t)$ to heat equation. We know that the only steady state solution is $u_p(x,t) = ax + b$ for some a, b. We will figure out a, b so that $u_p(0,t) = -1$ and $u_x(1,t) = 1$ (which are our non-homogeneous boundary conditions). Hence $u_p(0,t) = b = -1$ and $(u_p(x,t))_x = a$ which we want to be 1 when $x = 1$, i.e. $(u_p(x,t))_x = a = 1$. Hence we get $u_p(x,t) = x - 1$. We now let

$$v(x,t) = u(x,t) - u_p(x,t)$$

and hope that v will satisfy the heat equation with homogeneous boundary conditions. To see this, as u solves (1), and u_p is steady-state solution to heat equation, and heat equation is linear v solves the heat equation $v_t = 2v_{xx}$. Next, we check the boundary conditions

$$v(0,t) = u(0,t) - u_p(0,t) = -1 - (-1) = 0 \quad \text{and} \quad v_x(1,t) = u_x(1,t) - (u_p(1,t))_x = 1 - 1 = 0.$$

Hence v satisfies the homogeneous boundary conditions. We next see the initial condition

$$v(x,0) = u(x,0) - u_p(x,0) = x + \sin(\frac{3\pi x}{2}) - 1 - (x-1) = \sin(\frac{3\pi x}{2}).$$

If we summarize what we got for v is that

$$\begin{cases} v_t = 2v_{xx} & 0 \le x \le 1,\ t \ge 0 \\ v(0,t) = 0 & v_x(1,t) = 0 \\ v(x,0) = \sin(\frac{3\pi x}{2}). \end{cases}$$

From the first problem, we know that the general solution is (where $k = 2$, $L = 1$)

$$v(x,t) = \sum_{n=0}^{\infty} C_n \sin(\frac{(n+\frac{1}{2})\pi x}{1}) e^{-2((n+\frac{1}{2})\pi/1)^2 t}$$

and using this and the given initial condition for v we get

$$v(x,0) = \sum_{n=0}^{\infty} C_n \sin((n+\frac{1}{2})\pi x) = \sin(\frac{3\pi x}{2}).$$

From this we conclude that for $n = 1$ we have $C_1 \sin(\frac{3\pi x}{2})$ and therefore, $C_1 = 1$ and all other $C_n = 0$. Hence we have

$$v(x,t) = C_1 \sin(\frac{3\pi x}{2}) e^{-2(3\pi/2)^2 t} = \sin(\frac{3\pi x}{2}) e^{-2(3\pi/2)^2 t}.$$

As we put $v(x,t) = u(x,t) - u_p(x,t)$ and $u(x,t)$ is the function we are after which solves (1), we can get

$$u(x,t) = v(x,t) + u_p(x,t) = \sin(\frac{3\pi x}{2}) e^{-2(3\pi/(2))^2 t} + x - 1.$$

This is the solution to (1).

Question 3 (Exercise 3.4, 3) *Solve*

$$\begin{cases} u_t - u_{xx} = e^{-4t}\cos(t)\sin(2x), & 0 \le x \le \pi,\ t \ge 0, \\ u(0,t) = 0,\ \ u(\pi,t) = 0, \\ u(x,0) = \sin(3x). \end{cases} \tag{2}$$

Solution: Since the boundary conditions are homogeneous, we can pass to the second step. That is we shall look for where $u(x,t) = u_1(x,t) + u_2(x,t)$ where u_1 solves the homogeneous heat equation;

$$\begin{cases} (u_1)_t - (u_1)_{xx} = 0, & 0 \le x \le \pi,\ t \ge 0, \\ u_1(0,t) = 0,\ \ u_1(\pi,t) = 0, \\ u_1(x,0) = \sin(3x). \end{cases} \tag{3}$$

and u_2 solves the non-homogeneous heat equation with zero initial condition

$$\begin{cases} (u_2)_t - (u_2)_{xx} = e^{-4t}\cos(t)\sin(2x), & 0 \le x \le \pi,\ t \ge 0, \\ u_2(0,t) = 0,\ \ u_2(\pi,t) = 0, \\ u_2(x,0) = 0. \end{cases} \tag{4}$$

Then by linearity of the heat equation we conclude that $u(x,t) = u_1(x,t) + u_2(x,t)$ solves our original equation (2). We shall first focus on u_1, we know the general solution is (you can use the proposition from the book, or our lecture notes)

$$u_1(x,t) = \sum_{n_1}^{\infty} C_n e^{-n^2 t}\sin(nx)$$

and using the initial condition for u_1 we get

$$u_1(x,0) = \sin(3x) = \sum_{n_1}^{\infty} C_n \sin(nx)$$

which tells us $C_3 = 1$ and all other $C_n = 0$. Hence

$$u_1(x,t) = e^{-9t}\sin(3x).$$

solves (3). Now we focus on v_2. To solve (4), we shall use the Duhamel's principle. That is,

$$u_2(x,t) = \int_0^t \tilde{v}(x, t-s; s)ds$$

where $\tilde{v}$ solves

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = 0, & 0 \le x \le \pi,\ t \ge 0, \\ \tilde{v}(0,t;s) = 0,\ \ \tilde{v}(\pi,t;s) = 0, \\ \tilde{v}(x,0;s) = e^{-4s}\cos(s)\sin(2x). \end{cases} \tag{5}$$

Here you should think of $e^{-4s}\cos(s)$ as a constant independent of t. We know that the general solution is

$$\tilde{v}(x,t;s) = \sum_{n_1}^{\infty} C_n e^{-n^2 t}\sin(nx)$$

and using the initial condition in (5) we get

$$\tilde{v}(x,0;s) = \sum_{n_1}^{\infty} C_n \sin(nx) = e^{-4s}\cos(s)\sin(2x)$$

which tells us that $C_2 = e^{-4s}\cos(s)$ and all other $C_n = 0$. Hence we have (for $n = 2$)

$$\tilde{v}(x, t; s) = e^{-4s}\cos(s)e^{-4t}\sin(2x).$$

Using Duhamel's principle we have

$$u_2(x, t) = \int_0^t \tilde{v}(x, t-s; s)ds = \int_0^t e^{-4s}\cos(s)e^{-4(t-s)}\sin(2x)ds$$

To find u_2 we need to find that integral. After some algebra we see that

$$u_2(x, t) = \int_0^t e^{-4s}\cos(s)e^{-4(t-s)}\sin(2x)ds = e^{-4t}\sin(2x)\int_0^t \cos(s)ds = e^{-4t}\sin(2x)\sin(t)$$

Combining this with u_1 we get

$$u(x, t) = u_1(x, t) + u_2(x, t) = e^{-9t}\sin(3x) + e^{-4t}\sin(2x)\sin(t)$$

is the solution of (2).

Question 4 (Exercise 3.4, 4) *Solve*

$$\begin{cases} u_t - u_{xx} = t\cos(x) & 0 \le x \le \pi, \ t \ge 0 \\ u_x(0, t) = 0 & u_x(\pi, t) = 0 \\ u(x, 0) = 0. \end{cases} \tag{6}$$

Solution: Since the boundary conditions are zero, past to step two. Since the initial condition is zero, we can use the Duhamel's principle right away. That is, our solution is

$$u(x, t) = \int_0^t \tilde{v}(x, t-s; s)ds$$

where $\tilde{v}$ solves

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = 0 & 0 \le x \le \pi, \ t \ge 0 \\ \tilde{v}_x(0, t; s) = 0 & \tilde{v}_x(\pi, t; s) = 0 \\ \tilde{v}(x, 0; s) = s\cos(x). \end{cases}$$

We know from the last problem of HW4 that the general solution to this PDE is

$$\tilde{v}(x, t; s) = \sum_{n=1}^{\infty} C_n e^{-n^2 t}\cos(nx)$$

and using the initial condition for $\tilde{v}$ we get

$$\tilde{v}(x, 0; s) = \sum_{n=1}^{\infty} C_n\cos(nx) = s\cos(x)$$

From this we get $C_1 = s$ and all other $C_n = 0$. Hence (for $n = 1$)

$$\tilde{v}(x, t; s) = se^{-t}\cos(x).$$

As we know that

$$\begin{aligned} u(x, t) = \int_0^t \tilde{v}(x, t-s; s)ds = \int_0^t se^{-(t-s)}\cos(x)ds &= e^{-t}\cos(x)\int_0^t se^s ds \\ &= e^{-t}\cos(x)\int_0^t se^s ds = e^{-t}\cos(x)(te^t - e^t + 1) \\ &= \cos(x)(t - 1 + e^{-t}) \end{aligned}$$

is the solution of (6).

Question 5 (Exercise 3.4, 7) *Solve*

$$\begin{cases} u_t - u_{xx} = \frac{1}{\pi}xe^t + t[2 - \frac{2}{\pi}x + \sin(x)] & 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) = t^2 & u(\pi,t) = e^t \\ u(x,0) = \frac{x}{\pi} + \sin(2x). \end{cases} \tag{7}$$

Solution: As the boundary conditions are non-homogeneous, the first step is to make them homogeneous. To this end, we let

$$u_p(x,t) = (b(t) - a(t))x/L + a(t) = (\frac{e^t - t^2}{\pi})x + t^2$$

so that $u_p(0,t) = t^2$ and $u_p(\pi,t) = e^t$. The second step is to let

$$v(x,t) = u(x,t) - u_p(x,t)$$

so that the non-homogeneous boundary conditions become homogeneous. Now v solves the non-homogeneous heat equation

$$v_t - v_{xx} = u_t - u_{xx} - [(u_p)_t - (u_p)_{xx}] = \frac{1}{\pi}xe^t + t[2 - \frac{2}{\pi}x + \sin(x)] - [(\frac{e^t - 2t}{\pi})x + 2t - 0] = t\sin(x).$$

The boundary conditions

$$v(0,t) = u(0,t) - u_p(0,t) = t^2 - t^2 = 0 \quad \text{and} \quad v(\pi,t) = u(\pi,t) - u_p(\pi,t) = e^t - e^t = 0.$$

The initial condition

$$v(x,0) = u(x,0) - u_p(x,0) = \frac{x}{\pi} + \sin(2x) - \frac{x}{\pi} = \sin(2x).$$

Hence, combining all of these we see that v solves

$$\begin{cases} v_t - v_{xx} = t\sin(x) & 0 \le x \le \pi,\ t \ge 0, \\ v(0,t) = 0 & v(\pi,t) = 0, \\ v(x,0) = \sin(2x). \end{cases} \tag{8}$$

As v solves the non-homogeneous heat equation with initial conditions, the next step is to look for v_1, v_2 with $v(x,t) = v_1(x,t) + v_2(x,t)$ where v_1 solves homogeneous heat equation with the initial condition in (8)

$$\begin{cases} (v_1)_t - (v_1)_{xx} = 0 & 0 \le x \le \pi,\ t \ge 0 \\ v_1(0,t) = 0 & v_1(\pi,t) = 0 \\ v_1(x,0) = \sin(2x). \end{cases} \tag{9}$$

and v_2 solves the non-homogeneous heat equation with zero initial condition

$$\begin{cases} (v_2)_t - (v_2)_{xx} = t\sin(x) & 0 \le x \le \pi,\ t \ge 0 \\ v_2(0,t) = 0 & v_2(\pi,t) = 0 \\ v_2(x,0) = 0. \end{cases} \tag{10}$$

We first focus on v_1. We know the general solution is

$$v_1(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx).$$

Using this and the given initial condition for v_1 we have

$$v_1(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) = \sin(2x)$$

which tells us that $C_2 = 1$ and all other $C_n = 0$. Hence we have

$$v_1(x,t) = e^{-4t}\sin(2x).$$

We now focus on v_2. From Duhamel's principle

$$v_2(x,t) = \int_0^t \tilde{v}(x,t-s;s)ds$$

where $\tilde{v}(x,t;s)$ solves the following homogeneous heat equation

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = 0 & 0 \leq x \leq \pi,\ t \geq 0 \\ \tilde{v}(0,t;s) = 0 & \tilde{v}(\pi,t;s) = 0 \\ \tilde{v}(x,0;s) = s\sin(x). & \end{cases}$$

We know that the general solutions is

$$\tilde{v}(x,t;s) = \sum_{n=1}^{\infty} C_n e^{-n^2 t}\sin(nx).$$

Using this and the initial condition for $\tilde{v}(x,t;s)$ we get

$$\tilde{v}(x,0;s) = s\sin(x) = \sum_{n=1}^{\infty} C_n \sin(nx)$$

From this, we see that $C_1 = s$ and all other $C_n = 0$. Hence

$$\tilde{v}(x,t;s) = se^{-t}\sin(x).$$

Using this we get

$$\begin{aligned} v_2(x,t) &= \int_0^t \tilde{v}(x,t-s;s)ds \\ &= \int_0^t se^{-(t-s)}\sin(x)ds \\ &= e^{-t}\sin(x)\int_0^t se^s ds \\ &= e^{-t}\sin(x)[te^t - e^t + 1] \\ &= \sin(x)[t - 1 + e^- t]. \end{aligned}$$

Hence

$$v(x,t) = v_1(x,t) + v_2(x,t) = e^{-4t}\sin(2x) + \sin(x)(t - 1 + e^- t).$$

Finally,

$$u(x,t) = v(x,t) + u_p(x,t) = e^{-4t}\sin(2x) + \sin(x)(t - 1 + e^{-t}) + (\frac{e^t - t^2}{\pi})x + t^2$$

is the solution to (7).

Question 6 (Exercise 3.4, 8) *Solve*

$$\begin{cases} u_t - 4u_{xx} = e^t \sin(\frac{x}{2}) - \sin(t) & 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) = \cos(t) & u_x(\pi,t) = 0 \\ u(x,0) = 1. \end{cases} \tag{11}$$

Solution: As we have a non-homogeneous boundary condition, the first step is to make it homogeneous. That is, we need to find $u_p(x,t)$ such that

$$u_p(0,t) = \cos(t) \quad \text{and} \quad (u_p)_x(\pi,t) = 0.$$

To this end, we mimic the steady-state solution, but cheating a little. That is, let $u_p(x,t) = a(t)x + b(t)$ where we are gonna choose $a(t)$ and $b(t)$. Since we want $u_p(0,t) = \cos(t) = b(t)$. Hence we have $u_p(x,t) = a(t)x + \cos(t)$. On the other hand, $(u_p)_x(\pi,t) = a(t) = 0$, we get $a(t) = 0$. Hence $u_p(x,t) = \cos(t)$. The next step is to let

$$v(x,t) = u(x,t) - u_p(x,t) = u(x,t) - \cos(t).$$

From this we see that $v(x,t)$ satisfies the following non-homogeneous heat equation

$$v_t - 4v_{xx} = u_t - 4u_{xx} - [(u_p)_t - 4(u_p)_{xx}] = e^t \sin(\frac{x}{2}) - \sin(t) + \sin(t) = e^t \sin(\frac{x}{2}).$$

The boundary conditions becomes homogeneous

$$v(0,t) = u(0,t) - \cos(t) = \cos(t) - \cos(t) = 0 \quad \text{and} \quad v_x(x,t) = u_x(x,t) - 0 = 0 - 0 = 0.$$

Finally, the initial condition is

$$v(x,0) = u(x,0) - \cos(0) = 1 - 1 = 0.$$

Hence, v solves a non-homogeneous heat equation

$$\begin{cases} v_t - 4v_{xx} = e^t \sin(\frac{x}{2}) & 0 \le x \le \pi,\ t \ge 0 \\ v(0,t) = 0 & v_x(\pi,t) = 0 \\ v(x,0) = 0. \end{cases} \tag{12}$$

Since the initial condition is zero we can right away use the Duhamel's principle to find $v(x,t)$

$$v(x,t) = \int_0^t \tilde{v}(x, t-s; s) ds$$

where $\tilde{v}(x,t;s)$ solves the homogeneous Heat equation with homogeneous boundary conditions

$$\begin{cases} \tilde{v}_t - 4\tilde{v}_{xx} = 0 & 0 \le x \le \pi,\ t \ge 0 \\ \tilde{v}(0,t;s) = 0 & \tilde{v}_x(\pi,t;s) = 0 \\ \tilde{v}(x,0;s) = e^s \sin(\frac{x}{2}). \end{cases}$$

From the first problem, we know the general solution is (with $k = 4$, $L = 1$)

$$\tilde{v}(x,t;s) = \sum_{n=1}^{\infty} C_n \sin(\frac{nx}{2}) e^{-n^2 t}$$

Using the initial condition we can find C_n

$$\tilde{v}(x,0;s) = \sum_{n=1}^{\infty} C_n \sin(\frac{nx}{2}) = e^s \sin(\frac{x}{2})$$

which gives us that $C_1 = e^s$ and all other $C_n = 0$. Hence

$$\tilde{v}(x,t;s) = e^s \sin(\frac{x}{2})e^{-t}.$$

Using this we can $v(x,t)$

$$\begin{aligned} v(x,t) &= \int_0^t \tilde{v}(x,t-s;s)ds \\ &= \int_0^t e^s \sin(\frac{x}{2})e^{-(t-s)}ds \\ &= \sin(\frac{x}{2})e^{-t}\int_0^t e^{2s}ds \\ &= \sin(\frac{x}{2})e^{-t}\frac{1}{2}[e^2t-1] = \frac{1}{2}\sin(\frac{x}{2})[e^t - e^{-t}]. \end{aligned}$$

From this we can find the solution to (11)

$$u(x,t) = v(x,t) + u_p(x,t) = \frac{1}{2}\sin(\frac{x}{2})[e^t - e^{-t}] + \cos(t).$$