Question 1 (Exercise 3.3, 3) Find all product solutions of the heat equation

\[
\begin{cases}
  u_t = ku_{xx} & 0 \leq x \leq L, \ t \geq 0 \\
  u(0,t) = 0 & u_x(L,t) = 0 \\
  u(x,0) = f(x).
\end{cases}
\]

Solution: As the question ask, we will find all solutions of the form \( u(x,t) = X(x)T(t) \). We have

\[ u_t = X(x)T'(t) \quad \text{and} \quad U_{xx} = X''(x)T(t). \]

Using this in the Heat equation we get

\[ X(x)T'(t) = kX''(x)T(t) \quad \text{equivalently} \quad \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} \]

which can happen both ratios are constant. Say

\[ \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = c. \]

Then we obtain

\[ T'(t) - kcT(t) = 0 \quad \text{and} \quad X''(x) - cX(x) = 0. \]

We will also rewrite the boundary conditions in terms of \( X, T \). Since \( u(0,t) = X(0)T(t) = 0 \) we get \( X(0) = 0 \) (otherwise, if we choose \( T(t) = 0 \) we get zero solution). Similarly, \( u_x(L,t) = X'(L)T(t) = 0 \) which gives us \( X'(L) = 0 \). Hence we have \( X(0) = 0 = X'(L) \).

We focus on \( X''(x) - cX(x) = 0 \) with the boundary conditions \( X(0) = 0 = X'(L) \).

When \( c = 0 \) we have \( X''(x) = 0 \) which gives us \( X(x) = ax + b \) for some constant \( a, b \). Using the boundary conditions \( X(0) = 0 = X'(L) \) we get \( b = 0 = a \). Hence no non-trivial solution is coming from \( c = 0 \).

When \( \lambda^2 = c > 0 \), for some \( \lambda > 0 \), we have \( X(x) = ae^{\lambda x} + be^{-\lambda x} \) for some constant \( a, b \). Using \( X(0) = 0 \) we get \( a = -b \) and using \( X'(L) = 0 \) we get

\[ X'(L) = a\lambda e^{\lambda L} + a\lambda e^{-\lambda L} = 0 \]

implies first \( a = 0 \) second \( a = -b = 0 \). Hence we also have trivial solution in this case.

When \( -\lambda^2 = c < 0 \) for some \( \lambda > 0 \) we then have \( X(x) = a\cos(\lambda x) + b\sin(\lambda x) \) for some constant \( a, b \). Since \( X(0) = 0 \) we get \( a = 0 \). Similarly, \( X'(L) = -b\lambda \cos(\lambda L) = 0 \), this can happen if \( \lambda L \) is multiple of \( \pi/2 = (n + 1/2)\pi \). Hence \( \lambda L = (n + 1/2)\pi \), hence \( \lambda = (n + 1/2)\pi/L \) for \( n = 0, 1, 2, \ldots \) which is our eigenvalue. Hence

\[ \lambda_n = \frac{(n + 1/2)\pi}{L} \quad \text{and corresponding eigenfunction} \quad X_n(x) = A_n \sin\left(\frac{(n + 1/2)\pi x}{L}\right), \quad n = 0, 1, 2, \ldots \]

for some constant \( A_n \). For this value of \( c = -\lambda^2 = -((n + 1/2)\pi/L)^2 \) we solve \( 0 = T'(t) - kcT(t) = T'(t) + k((n + 1/2)\pi/L)^2 T(t) \) which has solution

\[ T_n(t) = B_n e^{-k((n + 1/2)\pi/L)^2 t}, \quad n = 0, 1, 2, \ldots \]

for some constant \( B_n \). Hence our solution is

\[ u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_nB_n \sin\left(\frac{(n + 1/2)\pi x}{L}\right)e^{-k((n + 1/2)\pi/L)^2 t} \]
For simplicity call $A_nB_n = C_n$. Now we shall use the initial condition to find $C_n$.

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} C_n \sin(\frac{(n + \frac{1}{2})\pi x}{L}) = C_0 \sin(\frac{\pi x}{2L}) + C_1 \sin(\frac{3\pi x}{2L}) + C_2 \sin(\frac{5\pi x}{2L}) + \ldots$$

where $C_0, C_1, C_2, \ldots$ are constant to be found. This will give us the solution.

**Question 2 (Exercise 3.3, 4)** Find all product solutions of the heat equation

\[
\begin{align*}
\begin{cases}
    u_t = 2u_{xx} & \quad 0 \leq x \leq 1, \; t \geq 0 \\
    u(0, t) = -1 & \quad u_x(1, t) = 1 \\
    u(x, 0) = x + \sin(\frac{3\pi x}{2}) - 1.
\end{cases}
\end{align*}
\]

(1)

**Solution:** We first should make the non-homogeneous boundary conditions homogeneous. To this end, we look for time-independent or steady-state solution $u_p(x, t)$ to heat equation. We know that the only steady state solution is $u_p(x,t) = ax + b$ for some $a , b$. We will figure out $a , b$ so that $u_p(0,t) = -1$ and $u_x(1,t) = 1$ (which are our non-homogeneous boundary conditions). Hence $u_p(0,t) = b = -1$ and $(u_p(x,t))_x = a$ which we want to be 1 when $x = 1$, i.e. $(u_p(x,t))_x = a = 1$. Hence we get $u_p(x,t) = x - 1$. We now let

$$v(x,t) = u(x,t) - u_p(x,t)$$

and hope that $v$ will satisfy the heat equation with homogeneous boundary conditions. To see this, as $u$ solves (1), and $u_p$ is steady-state solution to heat equation, and heat equation is linear $v$ solves the heat equation $v_t = 2v_{xx}$. Next, we check the boundary conditions

$$v(0,t) = u(0,t) - u_p(0,t) = -1 - (-1) = 0 \quad \text{and} \quad v_x(1,t) = u_x(1,t) - (u_p(1,t))_x = 1 - 1 = 0.$$

Hence $v$ satisfies the homogeneous boundary conditions. We next see the initial condition

$$v(x,0) = u(x,0) - u_p(x,0) = x + \sin(\frac{3\pi x}{2}) - 1 - (x - 1) = \sin(\frac{3\pi x}{2}).$$

If we summarize what we got for $v$ is that

\[
\begin{align*}
\begin{cases}
    v_t = 2v_{xx} & \quad 0 \leq x \leq 1, \; t \geq 0 \\
    v(0, t) = 0 & \quad v_x(1, t) = 0 \\
    v(x, 0) = \sin(\frac{3\pi x}{2}).
\end{cases}
\end{align*}
\]

From the first problem, we know that the general solution is (where $k = 2, L = 1$)

$$v(x,t) = \sum_{n=0}^{\infty} C_n \sin(\frac{(n + \frac{1}{2})\pi x}{L}) e^{-2((n + \frac{1}{2})\pi/2)^2 t}$$

and using this and the given initial condition for $v$ we get

$$v(x,0) = \sum_{n=0}^{\infty} C_n \sin(\frac{(n + \frac{1}{2})\pi x}{L}) = \sin(\frac{3\pi x}{2}).$$

From this we conclude that for $n = 1$ we have $C_1 \sin(\frac{3\pi x}{2})$ and therefore, $C_1 = 1$ and all other $C_n = 0$. Hence we have

$$v(x,t) = C_1 \sin(\frac{3\pi x}{2}) e^{-2(3\pi/2)^2 t} = \sin(\frac{3\pi x}{2}) e^{-2(3\pi/2)^2 t}.$$

As we put $v(x,t) = u(x,t) - u_p(x,t)$ and $u(x,t)$ is the function we are after which solves (1), we can get

$$u(x,t) = v(x,t) + u_p(x,t) = \sin(\frac{3\pi x}{2}) e^{-2(3\pi/2)^2 t} + x - 1.$$

This is the solution to (1).
Question 3 (Exercise 3.4, 3) Solve

\[
\begin{cases}
  u_t - u_{xx} = e^{-4t} \cos(t) \sin(2x), & 0 \leq x \leq \pi, \ t \geq 0, \\
  u(0, t) = 0, \ u(\pi, t) = 0, \\
  u(x, 0) = \sin(3x).
\end{cases}
\] (2)

Solution: Since the boundary conditions are homogeneous, we can pass to the second step. That is we shall look for where \( u(x, t) = u_1(x, t) + u_2(x, t) \) where \( u_1 \) solves the homogeneous heat equation;

\[
\begin{cases}
  (u_1)_t - (u_1)_{xx} = 0, & 0 \leq x \leq \pi, \ t \geq 0, \\
  u_1(0, t) = 0, \ u_1(\pi, t) = 0, \\
  u_1(x, 0) = \sin(3x).
\end{cases}
\] (3)

and \( u_2 \) solves the non-homogeneous heat equation with zero initial condition

\[
\begin{cases}
  (u_2)_t - (u_2)_{xx} = e^{-4t} \cos(t) \sin(2x), & 0 \leq x \leq \pi, \ t \geq 0, \\
  u_2(0, t) = 0, \ u_2(\pi, t) = 0, \\
  u_2(x, 0) = 0.
\end{cases}
\] (4)

Then by linearity of the heat equation we conclude that \( u(x, t) = u_1(x, t) + u_2(x, t) \) solves our original equation (2). We shall first focus on \( u_1 \), we know the general solution is (you can use the proposition from the book, or our lecture notes)

\[
u_1(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx)
\]

and using the initial condition for \( u_1 \) we get

\[
u_1(x, 0) = \sin(3x) = \sum_{n=1}^{\infty} C_n \sin(nx)
\]

which tells us \( C_3 = 1 \) and all other \( C_n = 0 \). Hence

\[
u_1(x, t) = e^{-9t} \sin(3x).
\]
solves (3). Now we focus on \( v_2 \). To solve (4), we shall use the Duhamel’s principle. That is,

\[
u_2(x, t) = \int_0^t \tilde{v}(x, t - s; s) ds
\]

where \( \tilde{v} \) solves

\[
\begin{cases}
  \tilde{v}_t - \tilde{v}_{xx} = 0, & 0 \leq x \leq \pi, \ t \geq 0, \\
  \tilde{v}(0, t; s) = 0, \ \tilde{v}(\pi, t; s) = 0, \\
  \tilde{v}(x, 0; s) = e^{-4s} \cos(s) \sin(2x).
\end{cases}
\] (5)

Here you should think of \( e^{-4s} \cos(s) \) as a constant independent of \( t \). We know that the general solution is

\[
\tilde{v}(x, t; s) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx)
\]

and using the initial condition in (5) we get

\[
\tilde{v}(x, 0; s) = \sum_{n=1}^{\infty} C_n \sin(nx) = e^{-4s} \cos(s) \sin(2x)
\]
which tells us that $C_2 = e^{-4s} \cos(s)$ and all other $C_n = 0$. Hence we have (for $n = 2$)
\[
\tilde{\varphi}(x, t; s) = e^{-4s} \cos(s) e^{-4t} \sin(2x).
\]

Using Duhamel’s principle we have
\[
u_2(x, t) = \int_0^t \tilde{\varphi}(x, t - s; s) ds = \int_0^t e^{-4s} \cos(s) e^{-4(t-s)} \sin(2x) ds
\]

To find $u_2$ we need to find that integral. After some algebra we see that
\[
u_2(x, t) = \int_0^t e^{-4s} \cos(s) e^{-4(t-s)} \sin(2x) ds = e^{-4t} \sin(2x) \int_0^t \cos(s) ds = e^{-4t} \sin(2x) \sin(t)
\]
Combining this with $u_1$ we get
\[
u(x, t) = u_1(x, t) + u_2(x, t) = e^{-9t} \sin(3x) + e^{-4t} \sin(2x) \sin(t)
\]
is the solution of (2).

**Question 4 (Exercise 3.4, 4) Solve**
\[
\begin{aligned}
u_t - \nu_{xx} &= t \cos(x) & 0 \leq x \leq \pi, \ t \geq 0 \\
u_x(0, t) &= 0 & u_x(\pi, t) = 0 \\
u(x, 0) &= 0.
\end{aligned}
\] (6)

**Solution:** Since the boundary conditions are zero, past to step two. Since the initial condition is zero, we can use the Duhamel’s principle right away. That is, our solution is
\[
u(x, t) = \int_0^t \tilde{\varphi}(x, t - s; s) ds
\]
where $\tilde{\varphi}$ solves
\[
\begin{aligned}
\tilde{\varphi}_t - \tilde{\varphi}_{xx} &= 0 & 0 \leq x \leq \pi, \ t \geq 0 \\
\tilde{\varphi}_x(0, t; s) &= 0 & \tilde{\varphi}_x(\pi, t; s) = 0 \\
\tilde{\varphi}(x, 0; s) &= s \cos(x).
\end{aligned}
\]
We know from the last problem of HW4 that the general solution to this PDE is
\[
\tilde{\varphi}(x, t; s) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \cos(nx)
\]
and using the initial condition for $\tilde{\varphi}$ we get
\[
\tilde{\varphi}(x, 0; s) = \sum_{n=1}^{\infty} C_n \cos(nx) = s \cos(x)
\]
From this we get $C_1 = s$ and all other $C_n = 0$. Hence (for $n = 1$)
\[
\tilde{\varphi}(x, t; s) = s e^{-t} \cos(x).
\]
As we know that
\[
u(x, t) = \int_0^t \tilde{\varphi}(x, t - s; s) ds = \int_0^t s e^{-(t-s)} \cos(x) ds = e^{-t} \cos(x) \int_0^t s e^s ds
\]
\[
= e^{-t} \cos(x) \int_0^t s e^s ds = e^{-t} \cos(x) (te^t - e^t + 1)
\]
\[
= \cos(x) (t - 1 + e^{-t})
\]
is the solution of (6).
Question 5 (Exercise 3.4, 7) Solve

\[ \begin{cases} u_t - u_{xx} = \frac{1}{\pi} xe^t + t[2 - \frac{2}{\pi} x + \sin(x)] & 0 \leq x \leq \pi, \ t \geq 0 \\ u(0, t) = t^2 \\ u(x, 0) = \frac{x}{\pi} + \sin(2x). \end{cases} \]  \tag{7} \]

Solution: As the boundary conditions are non-homogeneous, the first step is to make them homogeneous. To this end, we let

\[ u_p(x, t) = (b(t) - a(t))x/L + a(t) = \left(\frac{e^t - t^2}{\pi}\right)x + t^2 \]

so that \( u_p(0, t) = t^2 \) and \( u_p(\pi, t) = e^t \). The second step is to let

\[ v(x, t) = u(x, t) - u_p(x, t) \]

so that the non-homogeneous boundary conditions become homogeneous. Now \( v \) solves the non-homogeneous heat equation

\[ v_t - v_{xx} = u_t - u_{xx} - [(u_p)_t - (u_p)_{xx}] = \frac{1}{\pi} xe^t + t[2 - \frac{2}{\pi} x + \sin(x)] - \left[\frac{(e^t - t^2)}{\pi}\right] x + 2t - 0 = t \sin(x). \]

The initial condition

\[ v(x, 0) = u(x, 0) - u_p(x, 0) = \frac{x}{\pi} + \sin(2x) - \frac{x}{\pi} = \sin(2x). \]

Hence, combining all of these we see that \( v \) solves

\[ \begin{cases} v_t - v_{xx} = t \sin(x) & 0 \leq x \leq \pi, \ t \geq 0, \\ v(0, t) = 0 \\ v(\pi, t) = 0, \end{cases} \]

As \( v \) solves the non-homogeneous heat equation with initial conditions, the next step is to look for \( v_1, v_2 \) with \( v(x, t) = v_1(x, t) + v_2(x, t) \) where \( v_1 \) solves homogeneous heat equation with the initial condition in (8)

\[ \begin{cases} (v_1)_t - (v_1)_{xx} = 0 & 0 \leq x \leq \pi, \ t \geq 0 \\ v_1(0, t) = 0 \\ v_1(\pi, t) = 0, \end{cases} \]  \tag{9} \]

and \( v_2 \) solves the non-homogeneous heat equation with zero initial condition

\[ \begin{cases} (v_2)_t - (v_2)_{xx} = t \sin(x) & 0 \leq x \leq \pi, \ t \geq 0 \\ v_2(0, t) = 0 \\ v_2(\pi, t) = 0. \end{cases} \]  \tag{10} \]

We first focus on \( v_1 \). We know the general solution is

\[ v_1(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx). \]
Using this and the given initial condition for \( v_1 \) we have

\[
v_1(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) = \sin(2x)
\]

which tells us that \( C_2 = 1 \) and all other \( C_n = 0 \). Hence we have

\[
v_1(x,t) = e^{-4t} \sin(2x).
\]

We now focus on \( v_2 \). From Duhamel’s principle

\[
v_2(x,t) = \int_0^t \tilde{v}(x,t-s;s)ds
\]

where \( \tilde{v}(x,t;s) \) solves the following homogeneous heat equation

\[
\begin{cases}
\tilde{v}_t - \tilde{v}_{xx} = 0 & 0 \leq x \leq \pi, \; t \geq 0 \\
\tilde{v}(0,t;s) = 0 & \tilde{v}(\pi,t;s) = 0 \\
\tilde{v}(x,0;s) = s \sin(x).
\end{cases}
\]

We know that the general solutions is

\[
\tilde{v}(x,t;s) = \sum_{n=1}^{\infty} C_n e^{-n^2t} \sin(nx).
\]

Using this and the initial condition for \( \tilde{v}(x,t;s) \) we get

\[
\tilde{v}(x,0;s) = s \sin(x) = \sum_{n=1}^{\infty} C_n \sin(nx)
\]

From this, we see that \( C_1 = s \) and all other \( C_n = 0 \). Hence

\[
\tilde{v}(x,t;s) = se^{-t} \sin(x).
\]

Using this we get

\[
v_2(x,t) = \int_0^t \tilde{v}(x,t-s;s)ds
\]

\[
= \int_0^t se^{-(t-s)} \sin(x)ds
\]

\[
= e^{-t} \sin(x) \int_0^t se^{s} ds
\]

\[
= e^{-t} \sin(x)[te^t - e^t + 1]
\]

\[
= \sin(x)[t - 1 + e^{-t}].
\]

Hence

\[
v(x,t) = v_1(x,t) + v_2(x,t) = e^{-4t} \sin(2x) + \sin(x)(t - 1 + e^{-t}).
\]

Finally,

\[
u(x,t) = v(x,t) + u_p(x,t) = e^{-4t} \sin(2x) + \sin(x)(t - 1 + e^{-t}) + \left(\frac{e^t - \frac{t^2}{\pi}}{\pi}\right)x + t^2
\]

is the solution to (7).
Question 6 (Exercise 3.4, 8) Solve

\[
\begin{aligned}
&u_t - 4u_{xx} = e^t \sin\left(\frac{x}{2}\right) - \sin(t) \quad 0 \leq x \leq \pi, \quad t \geq 0 \\
u(0, t) = \cos(t) &\quad u_x(\pi, t) = 0 \\
u(x, 0) = 1.
\end{aligned}
\] (11)

**Solution:** As we have a non-homogeneous boundary condition, the first step is to make it homogeneous. That is, we need to find \( u_p(x, t) \) such that

\[
u(0, t) = \cos(t) \quad \text{and} \quad (u_p)_x(\pi, t) = 0.\]

To this end, we mimic the steady-state solution, but cheating a little. That is, let \( u_p(x, t) = a(t)x + b(t) \) where we are gonna choose \( a(t) \) and \( b(t) \). Since we want \( u_p(0, t) = \cos(t) = b(t) \). Hence we have \( u_p(x, t) = a(t)x + \cos(t) \). On the other hand, \( (u_p)_x(\pi, t) = a(t) = 0 \), we get \( a(t) = 0 \). Hence \( u_p(x, t) = \cos(t) \). The next step is to let

\[
v(x, t) = u(x, t) - u_p(x, t) = u(x, t) - \cos(t).
\]

From this we see that \( v(x, t) \) satisfies the following non-homogeneous heat equation

\[
v_t - 4v_{xx} = u_t - 4u_{xx} - [(u_p)_t - 4(u_p)_{xx}] = e^t \sin\left(\frac{x}{2}\right) - \sin(t) + \sin(t) = e^t \sin\left(\frac{x}{2}\right).
\]

The boundary conditions becomes homogeneous

\[
v(0, t) = u(0, t) - \cos(t) = \cos(t) - \cos(t) = 0 \quad \text{and} \quad v_x(x, t) = u_x(x, t) - 0 = 0 - 0 = 0.
\]

Finally, the initial condition is

\[
v(x, 0) = u(x, 0) - \cos(0) = 1 - 1 = 0.
\]

Hence, \( v \) solves a non-homogeneous heat equation

\[
\begin{aligned}
v_t - 4v_{xx} &= e^t \sin\left(\frac{x}{2}\right) \quad 0 \leq x \leq \pi, \quad t \geq 0 \\
v(0, t) &= 0 \quad v_x(\pi, t) = 0 \\
v(x, 0) &= 0.
\end{aligned}
\] (12)

Since the initial condition is zero we can right away use the Duhamel’s principle to find \( v(x, t) \)

\[
v(x, t) = \int_0^t \vartheta(x, t - s; s)ds
\]

where \( \vartheta(x, t; s) \) solves the homogeneous Heat equation with homogeneous boundary conditions

\[
\begin{aligned}
\vartheta_t - 4\vartheta_{xx} &= 0 \quad 0 \leq x \leq \pi, \quad t \geq 0 \\
\vartheta(0, t; s) &= 0 \quad \vartheta_x(\pi, t; s) = 0 \\
\vartheta(x, 0; s) &= e^s \sin\left(\frac{x}{2}\right).
\end{aligned}
\]

From the first problem, we know the general solution is (with \( k = 4, L = 1 \))

\[
\vartheta(x, t; s) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{nx}{2}\right)e^{-nt}
\]

Using the initial condition we can find \( C_n \)

\[
\vartheta(x, 0; s) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{nx}{2}\right) = e^s \sin\left(\frac{x}{2}\right)
\]
which gives us that $C_1 = e^s$ and all other $C_n = 0$. Hence

$$\tilde{v}(x, t; s) = e^s \sin\left(\frac{x}{2}\right) e^{-t}.$$ 

Using this we can $v(x, t)$

$$v(x, t) = \int_0^t \tilde{v}(x, t - s; s) ds$$

$$= \int_0^t e^s \sin\left(\frac{x}{2}\right) e^{-(t-s)} ds$$

$$= \sin\left(\frac{x}{2}\right) e^{-t} \int_0^t e^{2s} ds$$

$$= \sin\left(\frac{x}{2}\right) e^{-t} \frac{1}{2} [e^{2t} - 1] = \frac{1}{2} \sin\left(\frac{x}{2}\right) [e^t - e^{-t}].$$

From this we can find the solution to (11)

$$u(x, t) = v(x, t) + u_p(x, t) = \frac{1}{2} \sin\left(\frac{x}{2}\right) [e^t - e^{-t}] + \cos(t).$$