UCONN - Math 3435 - Spring 2018 - Problem set 6

Question 1 (Exercise 4.1, 2a) *Using the trigonometric identities, find the Fourier series of*

$$f(x) = \cos^2(\pi x)\sin^2(\pi x) \quad \textit{when} \quad -1 \le x \le 1.$$

Solution: Since

$$\cos^2(\theta) = \frac{1}{2}(\cos(2\theta)+1) \quad \text{and} \quad \sin^2(\theta) = 1-\cos^2(\theta)$$

Using these with $\theta = \pi x$ we get

$$\begin{aligned}
f(x) &= \cos^2(\pi x)\sin^2(\pi x) \\
&= \cos^2(\pi x)[1-\cos^2(\pi x)] \\
&= \cos^2(\pi x) - \cos^4(\pi x) \\
&= \frac{1}{2}(\cos(2\pi x)+1) - (\frac{1}{2}[\cos(2\pi x)+1))^2 \\
&= \frac{1}{2}\cos(2\pi x) + \frac{1}{2} - \frac{\cos^2(2\pi x)}{4} - \frac{\cos(2\pi x)}{2} - \frac{1}{4} \\
&= \frac{1}{2}\cos(2\pi x) + \frac{1}{2} - \frac{\cos(4\pi x)}{8} - \frac{1}{8} - \frac{\cos(2\pi x)}{2} - \frac{1}{4} \\
&= \frac{1}{8} - \frac{1}{8}\cos(4\pi x)
\end{aligned}$$

which is the Fourier series $\mathcal{F}(x)$ of $f(x)$.

Question 2 (Exercise 4.1, 2b) *Using the trigonometric identities, find the Fourier series of*

$$f(x) = \sin(x)[\sin(x)+\cos(x)]^2 \quad \textit{when} \quad -\pi \le x \le \pi.$$

Solution: We again use the above identities to get (additionally, $\sin^2(\theta)+\cos^2(\theta) = 1$ and $\sin(2x) = 2\sin(x)\cos(x)$)

$$\begin{aligned}
f(x) &= \sin(x)[\sin(x)+\cos(x)]^2 = \sin(x)[\sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x)] \\
&= \sin(x)[1+\sin(2x)] \\
&= \sin(x) + \sin(x)\sin(2x).
\end{aligned}$$

Moreover, we also use

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

with $\alpha = x$ and $\beta = 2x$ to get

$$f(x) = \frac{1}{2}\cos(x) + \sin(x) - \frac{1}{2}\cos(3x).$$

Question 3 (Exercise 4.1, 4a) *Find the Fourier series of $f(x)$ on $[-L, L]$*

$$f(x) = \begin{cases} 1 & \textit{when } 0 \le x \le L, \\ 0 & \textit{when } -L \le x < 0. \end{cases}$$

Solution: Notice that the function is neither even nor odd. Hence we have to find all the terms. We start with a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} 1dx = 1.$$

Then a_n, $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx &= \frac{1}{L}\int_{0}^{L} 1\cos(\frac{n\pi x}{L})dx \\
&= \frac{1}{L}\frac{\sin(\frac{n\pi x}{L})}{\frac{n\pi}{L}}|_0^L \\
&= \frac{1}{L}\frac{\sin(n\pi)}{\frac{n\pi}{L}} - \frac{1}{L}\frac{1}{\frac{n\pi}{L}} \\
&= \frac{\sin(n\pi)}{n\pi} - \frac{0}{n\pi} = 0.
\end{aligned}$$

We next find b_n, $n = 1, 2, \ldots,$

$$\begin{aligned}
b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx &= \frac{1}{L}\int_{0}^{L} 1\sin(\frac{n\pi x}{L})dx \\
&= \frac{1}{L}[-\frac{\cos(\frac{n\pi x}{L})}{\frac{n\pi}{L}}]|_0^L \\
&= -\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi} = -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{F}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})] \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n}\sin(\frac{n\pi x}{L})
\end{aligned}$$

Question 4 (Exercise 4.1, 4b) *Find the Fourier series of $f(x)$ on $[-L, L]$*

$$f(x) = |x| = \begin{cases} x & \textit{when } 0 \le x \le L, \\ -x & \textit{when } -L \le x < 0. \end{cases}$$

Notice that the function $f(x)$ is an even function. Therefore, the Fourier series of $f(x)$ will be a cosine series and all $b_n = 0$. We first find a_0 and then a_n. We also use the fact that if $G(x)$ is an even function on $[-L, L]$ then

$$\int_{-L}^{L} G(x)dx = 2\int_{0}^{L} G(x)dx.$$

Using this we have

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = 2\frac{1}{L}\int_{0}^{L} f(x)dx = 2\frac{1}{L}\int_{0}^{L} xdx = 2\frac{1}{L}\frac{x^2}{2}|_0^L = L.$$

Similarly, we have

$$\begin{aligned}
a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx &= 2\frac{1}{L}\int_0^L x\cos(\frac{n\pi x}{L})dx \\
&= \frac{2}{L}[\frac{x\sin(\frac{n\pi x}{L})}{\frac{n\pi}{L}}|_0^L - \frac{1}{\frac{n\pi}{L}}\int_0^L \sin(\frac{n\pi x}{L})dx] \\
&= \frac{2}{L}[\frac{L\sin(\frac{n\pi L}{L})}{\frac{n\pi}{L}} - 0 + \frac{\cos(\frac{n\pi x}{L})}{(\frac{n\pi}{L})^2}|_0^L] \\
&= \frac{2}{L}[0 - 0 + \frac{\cos(\frac{n\pi L}{L})}{(\frac{n\pi}{L})^2} - \frac{1}{(\frac{n\pi}{L})^2}] \\
&= \frac{2}{L}[\frac{(-1)^n}{(\frac{n\pi}{L})^2} - \frac{1}{(\frac{n\pi}{L})^2}].
\end{aligned}$$

Hence

$$a_n = \frac{2L}{\pi^2 n^2}((-1)^n - 1)$$

Hence

$$\begin{aligned}
\mathcal{F}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})] \\
&= \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{\pi^2 n^2}((-1)^n - 1)\cos(\frac{n\pi x}{L}).
\end{aligned}$$

Question 5 (Problem 2) *Let $f(x)$ be given as*

$$f(x) = \begin{cases} 0 & when\ -\pi \le x \le 0, \\ x & when\ 0 \le x < \pi. \end{cases}$$

1. *Find the Fourier series $\mathcal{F}(x)$ of $f(x)$ on $-\pi \le x \le \pi$.*
2. *Using the first part verify that*
$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}.$$
3. *Using the first part verify also that*
$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

Solution: We will first find the Fourier series of $f(x)$. We first find a_0;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{1}{\pi}\frac{x^2}{2}|_0^{\pi} = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}.$$

Then a_n, $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\frac{n\pi x}{\pi})dx &= \frac{1}{\pi}\int_0^{\pi} x\cos(nx)dx \\
&= \frac{1}{\pi}[\frac{x\sin(nx)}{n}|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx] \\
&= \frac{1}{\pi}[\frac{\pi\sin(\pi n)}{\pi} - 0 + \frac{1}{n^2}\cos(nx)|_0^{\pi}] \\
&= \frac{1}{\pi}[0 - 0 + \cos(\pi n) - \cos(0)] = \frac{1}{\pi}\frac{1}{n^2}((-1)^n - 1)
\end{aligned}$$

where we have used $\cos(\pi n) = (-1)^n$. Also when n is even we have $a_n = 0$. We next find b_n

$$
\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(\frac{n\pi x}{\pi})dx &= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)dx \\
&= \frac{1}{\pi}[-\frac{x\cos(nx)}{n}|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)dx \\
&= \frac{1}{\pi}[-\frac{\pi\cos(pin)}{n} - 0 + \frac{1}{n^2}\sin(nx)|_0^{\pi}] \\
&= \frac{1}{\pi}[-\frac{\pi(-1)^n}{n} - 0 + \frac{1}{n^2}[0-0]] = -\frac{1}{\pi}\frac{\pi(-1)^n}{n} \\
&= \frac{(-1)^{n+1}}{n}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathcal{F}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(\frac{n\pi x}{\pi} + b_n\sin(\frac{n\pi x}{\pi})] \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty}[\frac{1}{\pi}\frac{1}{n^2}((-1)^n - 1)\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)]
\end{aligned}
$$

Now to verify the first identity, we use (we will show this next section) that $\mathcal{F}(x) = f(x)$ on $-\pi < x < \pi$ and can choose $x = \pi/2$ to get

$$
\begin{aligned}
\frac{\pi}{2} &= f(\frac{\pi}{2}) \\
&= \mathcal{F}(\frac{\pi}{2}) = \frac{\pi}{4} + \sum_{n=1}^{\infty}[\frac{1}{\pi}\frac{1}{n^2}((-1)^n - 1)\cos(n\frac{\pi}{2}) + \frac{(-1)^{n+1}}{n}\sin(n\frac{\pi}{2})]
\end{aligned}
$$

Now we need to make couple observations; when $(-1)^n - 1 = 0$ when n is even. Also $\cos(n\frac{\pi}{2}) = 0$ when n is odd. Hence the cosine terms are all zero. We only left with sine term and the constant terms. Also, $\sin(n\frac{\pi}{2})$ is 1 or -1 when n is odd and it is 0 when n is even. Hence if we let $n = 2k+1$ we have $\sin(n\frac{\pi}{2}) = \sin((2k+1)\frac{\pi}{2})$ and it is 1 when k is even and it is -1 when k is odd. Hence,

$$
\sin(n\frac{\pi}{2}) = \sin((2k+1)\frac{\pi}{2}) = (-1)^k \quad k = 0, 1, 2, \ldots.
$$

Combining all of these we have

$$
\begin{aligned}
\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\frac{\pi}{2}) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^{(2k+1)+1}}{2k+1}(-1)^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.
\end{aligned}
$$

In order to verify the second identity, we choose $x = 0$ which is in $-\pi < x < \pi$ and use $\mathcal{F}(x) = f(x)$ to get

$$
\begin{aligned}
0 = f(0) = \mathcal{F}(0) &= \frac{\pi}{4} + \sum_{n=1}^{\infty}[\frac{1}{\pi}\frac{1}{n^2}((-1)^n - 1)\cos(0) + \frac{(-1)^{n+1}}{n}\sin(0)] \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{\pi}\frac{1}{n^2}((-1)^n - 1).
\end{aligned}
$$

Note that $(-1)^n - 1 = 0$ when n is even and it is -2 when n is odd. Therefore, if we let $n = 2k+1$ for $k = 0, 1, \ldots,$ we have

$$-\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{\pi} \frac{1}{n^2}((-1)^n - 1) = -2 \sum_{k=0}^{\infty} \frac{1}{\pi} \frac{1}{(2k+1)^2}.$$

which in turn gives us

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Question 6 (Problem 3) *Let $f(x)$ be given as*

$$f(x) = \begin{cases} 0 & \text{when } -\pi \le x < 0, \\ 1 & \text{when } 0 \le x < \pi. \end{cases}$$

- *Find the Fourier series $\mathcal{F}(x)$ of $f(x)$ on $-\pi \le x \le \pi$.*
- *Using the first part verify that*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$$

Solution: We use solution of Question 3 of this HW with $L = \pi$ to get

$$\begin{aligned} \mathcal{F}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})] \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n} \sin(\frac{n\pi x}{\pi}). \end{aligned}$$

Notice that $1 - (-1)^n = 0$ when n is even and it is 2 when n is odd. Hence if we replace $n = 2k+1$ we have

$$\mathcal{F}(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x).$$

Using the given hint $\mathcal{F}(x) = f(x)$ on $0 < x < \pi$ and choosing $x = \pi/2$ we get

$$1 = f(\frac{\pi}{2}) = \mathcal{F}(\frac{\pi}{2}) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)\frac{\pi}{2}).$$

After some algebra we get

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$