Question 1 (Exercise 4.1, 2a) Using the trigonometric identities, find the Fourier series of
\[ f(x) = \cos^2(\pi x) \sin^2(\pi x) \quad \text{when} \quad -1 \leq x \leq 1. \]

Solution: Since
\[ \cos^2(\theta) = \frac{1}{2}(\cos(2\theta) + 1) \quad \text{and} \quad \sin^2(\theta) = 1 - \cos^2(\theta) \]
Using these with \( \theta = \pi x \) we get
\[ f(x) = \cos^2(\pi x) \sin^2(\pi x) \]
\[ = \cos^2(\pi x)[1 - \cos^2(\pi x)] \]
\[ = \cos^2(\pi x) - \cos^4(\pi x) \]
\[ = \frac{1}{2}(\cos(2\pi x) + 1) - \left(\frac{1}{2}[\cos(2\pi x) + 1]\right)^2 \]
\[ = \frac{1}{2} \cos(2\pi x) + \frac{1}{2} - \frac{\cos^2(2\pi x)}{4} - \frac{\cos(2\pi x)}{2} - \frac{1}{4} \]
\[ = \frac{1}{2} \cos(2\pi x) + \frac{1}{2} - \frac{\cos(4\pi x)}{8} - \frac{1}{8} - \frac{\cos(2\pi x)}{2} - \frac{1}{4} \]
\[ = \frac{1}{8} - \frac{1}{8} \cos(4\pi x) \]
which is the Fourier series \( \mathcal{F}(x) \) of \( f(x) \).

Question 2 (Exercise 4.1, 2b) Using the trigonometric identities, find the Fourier series of
\[ f(x) = \sin(x)[\sin(x) + \cos(x)]^2 \quad \text{when} \quad -\pi \leq x \leq \pi. \]

Solution: We again use the above identities to get (additionally, \( \sin^2(\theta) + \cos^2(\theta) = 1 \) and \( \sin(2x) = 2\sin(x)\cos(x) \))
\[ f(x) = \sin(x)[\sin(x) + \cos(x)]^2 = \sin(x)[\sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x)] \]
\[ = \sin(x)[1 + \sin(2x)] \]
\[ = \sin(x) + \sin(x)\sin(2x). \]
Moreover, we also use \( \sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \)
with \( \alpha = x \) and \( \beta = 2x \) to get
\[ f(x) = \frac{1}{2} \cos(x) + \sin(x) - \frac{1}{2} \cos(3x). \]

Question 3 (Exercise 4.1, 4a) Find the Fourier series of \( f(x) \) on \([-L, L] \)
\[ f(x) = \begin{cases} 
1 & \text{when} \ 0 \leq x \leq L, \\
0 & \text{when} \ -L \leq x < 0.
\end{cases} \]
**Solution:** Notice that the function is neither even nor odd. Hence we have to find all the terms. We start with $a_0$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \frac{1}{L} \int_{0}^{L} 1 \, dx = 1.$$  

Then $a_n, n = 1, 2, \ldots$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n \pi x}{L}\right) \, dx = \frac{1}{L} \int_{0}^{L} 1 \cos\left(\frac{n \pi x}{L}\right) \, dx$$

$$= \frac{1}{L} \frac{\sin\left(\frac{n \pi x}{L}\right)}{\frac{n \pi}{L}} \bigg|_{0}^{L}$$

$$= \frac{1}{L} \frac{\sin(n \pi)}{n \pi} - \frac{1}{L} \frac{\sin(0)}{n \pi} = \frac{\sin(n \pi)}{n \pi} - 0 = 0.$$  

We next find $b_n, n = 1, 2, \ldots$,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n \pi x}{L}\right) \, dx = \frac{1}{L} \int_{0}^{L} 1 \sin\left(\frac{n \pi x}{L}\right) \, dx$$

$$= \frac{1}{L} \left[-\frac{\cos\left(\frac{n \pi x}{L}\right)}{\frac{n \pi}{L}}\right]_{0}^{L}$$

$$= -\frac{\cos(n \pi)}{n \pi} + \frac{1}{n \pi} = -\frac{(-1)^n}{n \pi} + \frac{1}{n \pi}.$$  

Hence

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n \pi x}{L}\right) + b_n \sin\left(\frac{n \pi x}{L}\right)\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{\pi n} \sin\left(\frac{n \pi x}{L}\right).$$

**Question 4 (Exercise 4.1, 4b)** Find the Fourier series of $f(x)$ on $[-L, L]$

$$f(x) = |x| = \begin{cases} x & \text{when } 0 \leq x \leq L, \\ -x & \text{when } -L \leq x < 0. \end{cases}$$

Notice that the function $f(x)$ is an even function. Therefore, the Fourier series of $f(x)$ will be a cosine series and all $b_n = 0$. We first find $a_0$ and then $a_n$. We also use the fact that if $G(x)$ is an even function on $[-L, L]$ then

$$\int_{-L}^{L} G(x) \, dx = 2 \int_{0}^{L} G(x) \, dx.$$  

Using this we have

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \frac{1}{L} \int_{0}^{L} f(x) \, dx = \frac{1}{L} \int_{0}^{L} x \, dx = \frac{1}{L} \frac{x^2}{2} \bigg|_{0}^{L} = L.$$
Similarly, we have

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 2 \frac{1}{L} \int_{0}^{L} x \cos\left(\frac{n\pi x}{L}\right) dx \]

\[ = \frac{2}{L} \left[ \frac{x \sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} \right]_0^L - \frac{1}{n\pi} \int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \]

\[ = \frac{2}{L} \left[ \frac{L \sin\left(\frac{n\pi L}{L}\right)}{\frac{n\pi}{L}} - 0 + \frac{\cos\left(\frac{n\pi L}{L}\right)}{\left(\frac{n\pi}{L}\right)^2} \right]_0^L \]

\[ = \frac{2}{L} \left[ 0 - 0 + \frac{\cos\left(\frac{n\pi L}{L}\right)}{\left(\frac{n\pi}{L}\right)^2} - \frac{1}{\left(\frac{n\pi}{L}\right)^2} \right] \]

\[ = \frac{2}{L} \left[ \frac{(-1)^n}{\left(\frac{n\pi}{L}\right)^2} - \frac{1}{\left(\frac{n\pi}{L}\right)^2} \right]. \]

Hence

\[ a_n = \frac{2L}{\pi^2 n^2}((-1)^n - 1). \]

Hence

\[ \mathcal{F}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)] \]

\[ = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{\pi^2 n^2}((-1)^n - 1) \cos\left(\frac{n\pi x}{L}\right). \]

**Question 5 (Problem 2)** Let \( f(x) \) be given as

\[ f(x) = \begin{cases} 
0 & \text{when } -\pi \leq x \leq 0, \\
\pi & \text{when } 0 \leq x < \pi.
\end{cases} \]

1. Find the Fourier series \( \mathcal{F}(x) \) of \( f(x) \) on \(-\pi \leq x \leq \pi\).

2. Using the first part verify that

\[ \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}. \]

3. Using the first part verify also that

\[ \frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \]

**Solution:** We will first find the Fourier series of \( f(x) \). We first find \( a_0 \);

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{0}^{\pi} x dx = \frac{1}{\pi} \frac{x^2}{2} \bigg|_{0}^{\pi} = \frac{1}{2} \frac{\pi^2}{2} = \frac{\pi}{2}. \]

Then \( a_n, n = 1, 2, \ldots, \)

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx = \frac{1}{\pi} \int_{0}^{\pi} x \cos(nx) dx \]

\[ = \frac{1}{\pi} \left[ \frac{x \sin(nx)}{n} \right]_{\pi}^{0} - \frac{1}{n} \int_{0}^{\pi} \sin(nx) dx \]

\[ = \frac{1}{\pi} \left[ \frac{\pi \sin(\pi n)}{n} \right] - 0 + \frac{1}{n^2} \cos(nx) \bigg|_{0}^{\pi} \]

\[ = \frac{1}{\pi} [0 - 0 + \cos(\pi n) - \cos(0)] = \frac{1}{\pi} \frac{(-1)^n}{n^2}((-1)^n - 1) \]
where we have used $\cos(\pi n) = (-1)^n$. Also when $n$ is even we have $a_n = 0$. We next find $b_n$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx = \frac{1}{\pi} \int_{0}^{\pi} x \sin(nx) dx$$

$$= \frac{1}{\pi} \left[ -\frac{x \cos(nx)}{n} \right]_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} \cos(nx) dx$$

$$= \frac{1}{\pi} \left[ -\frac{\pi \cos(n\pi)}{n} \right] - 0 + \frac{1}{n^2} \sin(n\pi)$$

$$= \frac{1}{\pi} \left[ -\frac{(\pi(-1)^n}{n} - 0 + \frac{1}{n^2}[0 - 0] \right] = -\frac{1}{\pi} \frac{(\pi(-1)^n}{n}$$

$$= (-1)^{n+1} \frac{n}{n}$$

Hence we have

$$\mathcal{F}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)]$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[ \frac{1}{\pi} \frac{1}{n^2} ((-1)^n - 1) \cos(n\pi) + \frac{(-1)^n+1}{n} \sin(n\pi) \right]$$

Now to verify the first identity, we use (we will show this next section) that $\mathcal{F}(x) = f(x)$ on $-\pi < x < \pi$ and can choose $x = \pi/2$ to get

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right)$$

$$\mathcal{F}\left(\frac{\pi}{2}\right) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[ \frac{1}{\pi} \frac{1}{n^2} ((-1)^n - 1) \cos(n\pi) + \frac{(-1)^n+1}{n} \sin(n\pi) \right]$$

Now we need to make couple observations; when $(-1)^n - 1 = 0$ when $n$ is even. Also $\cos(n\pi/2) = 0$ when $n$ is odd. Hence the cosine terms are all zero. We only left with sine term and the constant terms. Also, $\sin(n\pi/2)$ is 1 or $-1$ when $n$ is odd and it is 0 when $n$ is even. Hence if we let $n = 2k + 1$ we have $\sin(n\pi/2) = \sin((2k+1)\pi/2)$ and it is 1 when $k$ is even and it is $-1$ when $k$ is odd. Hence,

$$\sin(n\pi/2) = \sin((2k+1)\pi/2) = (-1)^k \quad k = 0, 1, 2, \ldots.$$  

Combining all of these we have

$$\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n} \sin(n\pi/2)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}+1}{2k+1} (-1)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

In order to verify the second identity, we choose $x = 0$ which is in $-\pi < x < \pi$ and use $\mathcal{F}(x) = f(x)$ to get

$$0 = f(0) = \mathcal{F}(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[ \frac{1}{\pi} \frac{1}{n^2} ((-1)^n - 1) \cos(0) + \frac{(-1)^n+1}{n} \sin(0) \right]$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi} \frac{1}{n^2} ((-1)^n - 1).$$
Note that \((-1)^n - 1 = 0\) when \(n\) is even and it is \(-2\) when \(n\) is odd. Therefore, if we let \(n = 2k + 1\) for \(k = 0, 1, \ldots\), we have
\[
-\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{\pi n^2}((-1)^n - 1) = -2 \sum_{k=0}^{\infty} \frac{1}{\pi (2k+1)^2}.
\]
which in turn gives us
\[
\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.
\]

**Question 6 (Problem 3)** Let \(f(x)\) be given as
\[
f(x) = \begin{cases} 
0 & \text{when } -\pi \leq x < 0, \\
1 & \text{when } 0 \leq x < \pi.
\end{cases}
\]

- Find the Fourier series \(F(x)\) of \(f(x)\) on \(-\pi \leq x \leq \pi\).
- Using the first part verify that
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.
\]

**Solution:** We use solution of Question 3 of this HW with \(L = \pi\) to get
\[
F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]
= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{\pi n} \sin\left(\frac{n\pi x}{\pi}\right).
\]
Notice that \(1 - (-1)^n = 0\) when \(n\) is even and it is \(2\) when \(n\) is odd. Hence if we replace \(n = 2k + 1\) we have
\[
F(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi (2k+1)} \sin((2k+1)x).
\]
Using the given hint \(F(x) = f(x)\) on \(0 < x < \pi\) and choosing \(x = \pi/2\) we get
\[
1 = f(\frac{\pi}{2}) = F(\frac{\pi}{2}) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi (2k+1)} \sin((2k+1)\frac{\pi}{2}).
\]
After some algebra we get
\[
\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k - 1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1}.
\]