UCONN - Math 3435 - Spring 2018 - Problem set 7

Question 1 (Exercise 5.2, 1a) *Find the solution of*

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le L,\ -\infty < t < \infty \\ u(0,t) = 0 = u(L,t) & -\infty < t < \infty \\ u(x,0) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L}) & 0 \le x \le L \\ u_t(x,0) = \frac{1}{2}\sin(\frac{2\pi x}{L}) & 0 \le x \le L. \end{cases}$$

Solution: From Proposition 1 we know that the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \sin(\frac{n\pi a t}{L}) + B_n \cos(\frac{n\pi a t}{L})] \sin(\frac{n\pi x}{L}).$$

Using given initial conditions we will find A_n and B_n. Using the first initial condition and the general solution we have

$$u(x,0) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L}) = \sum_{n=1}^{\infty} [A_n 0 + B_n \cos(0)] \sin(\frac{n\pi x}{L})$$

Hence we get

$$3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L}) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$$

From this we see that $B_1 = 3$ and $B_4 = -1$ and all other $B_n = 0$. We next use the second initial condition to find A_n. To this end, we first see

$$u_t(x,t) = \sum_{n=1}^{\infty} [A_n \frac{n\pi a}{L} \cos(\frac{n\pi a t}{L}) - B_n \frac{n\pi a}{L} \sin(\frac{n\pi a t}{L})] \sin(\frac{n\pi x}{L}).$$

Evaluating this at $t = 0$ we have

$$u_t(x,0) = \frac{1}{2}\sin(\frac{2\pi x}{L}) = \sum_{n=1}^{\infty} [A_n \frac{n\pi a}{L} \cos(0) - B_n \frac{n\pi a}{L} \sin(0)] \sin(\frac{n\pi x}{L}).$$

From this we get

$$\frac{1}{2}\sin(\frac{2\pi x}{L}) = \sum_{n=1}^{\infty} [A_n \frac{n\pi a}{L} \sin(\frac{n\pi x}{L}).$$

Hence $A_2 \frac{2\pi a}{L} = 1/2$ and all other $A_n = 0$. Hence $A_2 = L/(2n\pi a)$Combining all these, we get

$$\begin{aligned} u(x,t) &= A_2 \sin(\frac{2\pi a t}{L}) \sin(\frac{2\pi x}{L}) + B_1 \cos(\frac{\pi a t}{L}) \sin(\frac{n\pi x}{L}) + B_4 \cos(\frac{4\pi a t}{L}) \sin(\frac{4\pi x}{L}) \\ &= \frac{L}{4\pi a} \sin(\frac{2\pi a t}{L}) \sin(\frac{2\pi x}{L}) + 3\cos(\frac{\pi a t}{L}) \sin(\frac{\pi x}{L}) - \cos(\frac{4\pi a t}{L}) \sin(\frac{4\pi x}{L}) \end{aligned}$$

Question 2 (Exercise 5.2, 1c) *Find the solution of*

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le L,\ -\infty < t < \infty \\ u(0,t) = 0 = u(L,t) & -\infty < t < \infty \\ u(x,0) = 0 & 0 \le x \le L \\ u_t(x,0) = \sin(\frac{\pi x}{L}) \cos^2(\frac{\pi x}{L}) & 0 \le x \le L. \end{cases}$$

Solution: We first use the given hint that we need to rewrite $\sin(\frac{\pi x}{L})\cos^2(\frac{\pi x}{L})$ without the square. We can use the following trig identity

$$\cos^2(\frac{\pi x}{L}) = 1 - \sin^2(\frac{\pi x}{L}) \quad \text{and} \quad \sin^2(\frac{\pi x}{L}) = \frac{1}{2}[1 - \cos(\frac{2\pi x}{L})]$$

and

$$\sin(\frac{\pi x}{L})\cos(\frac{2\pi x}{L}) = \frac{1}{2}[\sin(\frac{\pi x}{L} + \frac{2\pi x}{L}) + \sin(\frac{\pi x}{L} - \frac{2\pi x}{L})$$

then

$$\begin{aligned}
\sin(\frac{\pi x}{L})\cos^2(\frac{\pi x}{L}) &= \sin(\frac{\pi x}{L}) - \sin^3(\frac{\pi x}{L}) = \sin(\frac{\pi x}{L}) - \sin^2(\frac{\pi x}{L})\sin(\frac{\pi x}{L}) \\
&= \sin(\frac{\pi x}{L}) - \sin(\frac{\pi x}{L})\frac{1}{2}[1 - \cos(\frac{2\pi x}{L})] \\
&= \frac{1}{2}\sin(\frac{\pi x}{L}) + \frac{1}{2}\sin(\frac{\pi x}{L})\cos(\frac{2\pi x}{L}) \\
&= \frac{1}{2}\sin(\frac{\pi x}{L}) + \frac{1}{4}[\sin(\frac{\pi x}{L} + \frac{2\pi x}{L}) + \sin(\frac{\pi x}{L} - \frac{2\pi x}{L})] \\
&= \frac{1}{4}\sin(\frac{\pi x}{L}) + \frac{1}{4}\sin(\frac{3\pi x}{L}).
\end{aligned}$$

Now we can solve the Wave equation. Once again from Proposition 1 we know that the general solution is

$$u(x,t) = \sum_{n=1}^{\infty}[A_n \sin(\frac{n\pi at}{L}) + B_n \cos(\frac{n\pi at}{L})]\sin(\frac{n\pi x}{L}).$$

To find A_n and B_n we use the initial conditions.

$$u(x,0) = 0 = \sum_{n=1}^{\infty}[A_n \sin(0) + B_n \cos(0)]\sin(\frac{n\pi x}{L}).$$

From this we get $B_n = 0$ for all $n = 1, 2, \ldots$. Similarly,

$$u_t(x,t) = \sum_{n=1}^{\infty}[A_n \frac{n\pi a}{L}\cos(\frac{n\pi at}{L}) - B_n \frac{n\pi a}{L}\sin(\frac{n\pi at}{L})]\sin(\frac{n\pi x}{L}).$$

Evaluating this at $t = 0$ we have

$$u_t(x,0) = \frac{1}{4}\sin(\frac{\pi x}{L}) + \frac{1}{4}\sin(\frac{3\pi x}{L}) = \sum_{n=1}^{\infty}[A_n \frac{n\pi a}{L}\cos(0) - B_n \frac{n\pi a}{L}\sin(0)]\sin(\frac{n\pi x}{L}).$$

Therefore

$$\frac{1}{4}\sin(\frac{\pi x}{L}) + \frac{1}{4}\sin(\frac{3\pi x}{L}) = \sum_{n=1}^{\infty}[A_n \frac{n\pi a}{L}\sin(\frac{n\pi x}{L})$$

From this we get $A_1 \frac{\pi a}{L} = 1/4$ and $A_3 \frac{3\pi a}{L} = 1/4$ and all other $A_n = 0$. That is, $A_1 = L/(4\pi a)$ and $A_3 = L/(12\pi a)$. Collecting all of these we get

$$\begin{aligned}
u(x,t) &= A_1 \sin(\frac{\pi at}{L})\sin(\frac{\pi x}{L}) + A_3 \sin(\frac{3\pi at}{L})\sin(\frac{3\pi x}{L}) \\
&= \frac{L}{4\pi a}\sin(\frac{\pi at}{L})\sin(\frac{\pi x}{L}) + \frac{L}{12\pi a}\sin(\frac{3\pi at}{L})\sin(\frac{3\pi x}{L}).
\end{aligned}$$

Question 3 (Exercise 5.2, 5a) *Let $v(x,t)$ and $w(x,t)$ be two C^2 solutions of the problem*

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le L, \ -\infty < t < \infty \\ u(0,t) = 0 = u(L,t) & -\infty < t < \infty. \end{cases}$$

Show that

$$\frac{d}{dt}\int_0^L [a^2 v_x(x,t) w_x(x,t) + v_t(x,t) w_t(x,t)] dx = 0.$$

Solution: We can take the derivative inside the integral as we have C^2 solutions. Therefore,

$$\begin{aligned} \frac{d}{dt}\int_0^L [a^2 v_x(x,t) w_x(x,t) + v_t(x,t) w_t(x,t)] dx &= \int_0^L \frac{d}{dt}[a^2 v_x(x,t) w_x(x,t) + v_t(x,t) w_t(x,t)] dx \\ &= \int_0^L [a^2 v_{xt} w_x + a^2 v_x w_{xt} + v_{tt} w_t + v_t w_{tt}] dx. \end{aligned}$$

Since v and w are solutions then we can replace v_{tt} by $a^2 v_{xx}$ and w_{tt} by $a^2 w_{xx}$ to get

$$\begin{aligned} \frac{d}{dt}\int_0^L [a^2 v_x(x,t) w_x(x,t) + v_t(x,t) w_t(x,t)] dx &= \int_0^L [a^2 v_{xt} w_x + a^2 v_x w_{xt} + v_{tt} w_t + v_t w_{tt}] dx \\ &= \int_0^L [a^2 v_{xt} w_x + a^2 v_x w_{xt} + a^2 v_{xx} w_t + v_t a^2 w_{xx}] dx \\ &= a^2 \int_0^L [v_{xt} w_x + v_x w_{xt} + v_{xx} w_t + v_t w_{xx}] dx \\ &= a^2 \int_0^L (v_t w_x + v_x w_t)_x dx \\ &= a^2 [v_t w_x + v_x w_t]_{x=0}^{x=L} \\ &= a^2 [v_t(L,t) w_x(L,t) + v_x(L,t) w_t(L,t) - v_t(0,t) w_x(0,t) - v_x(0,t) w_t(0,t)]. \end{aligned}$$

Notice that as v, w solve the Wave equation with boundary conditions we have $v(L,t) = 0$ which gives us $v_t(L,t) = 0$. Similarly, it is true for w; $w_t(L,t) = 0$. Likewise, $v(0,t) = 0$ which gives us $v_t(0,t) = 0$. Similarly, it is true for w; $w_t(0,t) = 0$. Using these above, we have

$$\begin{aligned} &\frac{d}{dt}\int_0^L [a^2 v_x(x,t) w_x(x,t) + v_t(x,t) w_t(x,t)] dx \\ &\quad = a^2 [v_t(L,t) w_x(L,t) + v_x(L,t) w_t(L,t) - v_t(0,t) w_x(0,t) - v_x(0,t) w_t(0,t)] = 0. \end{aligned}$$

Question 4 (Exercise 5.2, 9) *Use the separation of variables to find all product solutions of the problem (with $k > 0$))*

$$\begin{cases} u_{tt} = a^2 u_{xx} - k u_t & 0 \le x \le L, \ -\infty < t < \infty \\ u(0,t) = 0 = u(L,t) & -\infty < t < \infty. \end{cases}$$

for the string with air resistance and fixed ends.

Solution: Let $u(x,t) = X(x)T(t)$ be the separable solutions. Then

$$u_t = XT', \quad u_{tt} = XT'', \quad \text{and} \quad u_{xx} = X''T.$$

Note that we we can rewrite the equation $u_{tt} = a^2 u_{xx} - k u_t$ as $u_{tt} - a^2 u_{xx} + k u_t = 0$. Hence

$$0 = u_{tt} - a^2 u_{xx} + k u_t = XT'' - a^2 X'' T - kXT'.$$

Moving X's one side and T's other side we get

$$\frac{X''}{X} = \frac{T'' - kT'}{a^2T} = \lambda.$$

From this we get two differential equations

$$X'' - \lambda X = 0 \quad \text{and} \quad T'' - kT' - a^2\lambda T = 0.$$

We now rewrite the boundary conditions in terms of X and T.

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0.$$

From these we see that $X(0) = 0 = X(L)$. (Otherwise, we get the trivial solution). From the Heat part, we know that the equation with boundary conditions

$$X'' - \lambda X = 0 \quad \text{and} \quad X(0) = 0 = X(L)$$

has solutions only when $\lambda < 0$, say $\lambda = -m^2$. Indeed, we have $X'' - \lambda X = X'' + m^2X = 0$ and solutions are

$$X(x) = A\sin(mx) + B\cos(mx).$$

Using the boundary values we have

$$X(0) = 0 = B.$$

Hence $X(x) = A\sin(mx)$. Using the second boundary value we have

$$X(L) = 0 = A\sin(mL)$$

and this happens when $mL = n\pi$ for $n = 1, 2, \ldots$. Therefore, $m = n\pi/L$ and $\lambda = -m^2 = -(n\pi/L)^2$. Solutions are (we neglect the constant A as we will have constants coming from second equation corresponding to T as we did earlier while dealing with the Heat equation)

$$X_n(x) = \sin(\frac{n\pi x}{L}) \quad \text{for } n = 1, 2, \ldots.$$

We now return to differential equation for T and we will solve that equation when $\lambda = -m^2 = -(n\pi/L)^2$

$$0 = T'' - kT' - a^2\lambda T = T'' - kT' + \frac{a^2n^2\pi^2}{L^2}T.$$

The corresponding characteristic equation is

$$r^2 - kr + \frac{a^2n^2\pi^2}{L^2} = 0$$

and we shall find the roots of this equation.

$$(r - \frac{k}{2})^2 - \frac{k^2}{4} + \frac{a^2n^2\pi^2}{L^2} = 0$$

From this we get

$$r_{1,2} = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \frac{a^2n^2\pi^2}{L^2}}.$$

Hence solutions are

$$T(t) = A_ne^{r_1t} + B_ne^{r_2t}$$

Combining this with the solution correspond to X we get

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} [A_ne^{r_1t} + B_ne^{r_2t}]\sin(\frac{n\pi x}{L})$$

where r_1 and r_2 are roots we found above.

Here notice that $\frac{k^2}{4} - \frac{a^2n^2\pi^2}{L^2}$ will be negative for some n and in this case we have solutions sine and cosine but we still can write them as exponential by using a well-known identity.