UCONN - Math 3435 - Spring 2018 - Problem set 7

Question 1 (Exercise 5.2, 1a) *Find the solution of*

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty, \ -\infty < t < \infty \\ u(x,0) = x^2 & -\infty < x < \infty, \\ u_t(x,0) = x & -\infty < x < \infty. \end{cases}$$

Solution: We shall use the D'Alembert's formula to find the solution.

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr$$

where here $f(x) = x^2$ and $g(x) = x$. Hence we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}[(x+at)^2 + (x-at)^2] + \frac{1}{2a}\int_{x-at}^{x+at} rdr \quad = \frac{1}{2}[x^2 + 2xat + a^2t^2 + x^2 - 2xat + a^2t^2 + \frac{1}{2a}\frac{r^2}{2}|_{x-at}^{x+at} \\ &= x^2 + a^2t^2 + \frac{1}{4a}[(x+at)^2 - (x-at)^2] \\ &= x^2 + a^2t^2 + \frac{1}{4a}4axt \\ &= x^2 + a^2t^2 + xt. \end{aligned}$$

Question 2 (Exercise 5.2, 1c) *Find the solution of*

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty, \ -\infty < t < \infty \\ u(x,0) = 0 & -\infty < x < \infty, \\ u_t(x,0) = 1 & -\infty < x < \infty. \end{cases}$$

Solution: Here $f(x) = 0$ and $g(x) = 1$ where

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr.$$

Hence

$$u(x,t) = 0 + \frac{1}{2a}\int_{x-at}^{x+at} 1dr = \frac{1}{2a}r|_{x-at}^{x+at} = t.$$

Question 3 (Exercise 5.2, 1d) *Find the solution of*

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty, \ -\infty < t < \infty \\ u(x,0) = 1 & -\infty < x < \infty, \\ u_t(x,0) = 0 & -\infty < x < \infty. \end{cases}$$

Solution: Here $f(x) = 1$ and $g(x) = 0$ where

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr.$$

Hence

$$u(x,t) = \frac{1}{2}[1+1] + 0 = 1.$$

Question 4 (Exercise 5.2, 1e) *Find the solution of*

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty,\ -\infty < t < \infty \\ u(x,0) = \sin(x) & -\infty < x < \infty, \\ u_t(x,0) = a\cos(x) & -\infty < x < \infty. \end{cases}$$

Solution: Here $f(x) = \sin(x)$ and $g(x) = a\cos(x)$ where

$$u(x,t) = \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} a\cos(r)dr.$$

Hence

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} a\cos(r)dr \\ &= \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{1}{2}\sin(r)|_{x-at}^{x+at} \\ &= \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{1}{2}[\sin(x+at) - \sin(x-at)] \\ &= \sin(x+at). \end{aligned}$$

Question 5 (Exercise 5.2, 6) *Let $u(x,t)$ be a solution to*

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty,\ -\infty < t < \infty \\ u(x,0) = x^2 & -\infty < x < \infty, \\ u_t(x,0) = x & -\infty < x < \infty. \end{cases}$$

Here $f(x)$ is C^2 and $g(x)$ is C^1 and both vanish outside of $[-b,b]$ for some $b > 0$. Then show that

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2a}\int_{\infty}^{\infty} g(r)dr = \frac{1}{2a}\int_{-b}^{b} g(r)dr.$$

Solution: As $f \in C^2$ and $g \in C^1$ and defined on $-\infty < x < \infty$ then we can use D'Alambert's formula to get

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr.$$

Now we can take limit as $t \to \infty$

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}[\frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr].$$

We first observe that t is large enough so that $x - at < -b$ and $x + at > b$ then f vanishes. Hence we get

$$\lim_{t\to\infty} u(x,t) = 0 + 0 + \lim_{t\to\infty}\frac{1}{2a}\int_{x-at}^{x+at} g(r)dr = \frac{1}{2a}\int_{\infty}^{\infty} g(r)dr.$$

Since g also vanishes outside of $[-b,b]$ then integral is zero, therefore there is no contribution from those parts. What is left if

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2a}\int_{\infty}^{\infty} g(r)dr = \frac{1}{2a}\int_{-b}^{b} g(r)dr.$$

Question 6 (Exercise 5.2, 7) *Verify that the solution you found in HW7 Problem 1a (Section 5.1 exercise 1a)to the Wave equation*

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le L,\ -\infty < t < \infty \\ u(0,t) = 0 = u(L,t) & -\infty < t < \infty \\ u(x,0) = f(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L}) & 0 \le x \le L \\ u_t(x,0) = g(x) = \frac{1}{2}\sin(\frac{2\pi x}{L}) & 0 \le x \le L. \end{cases}$$

will agree with the solution you will get by using the D'Alembert's formula.

Solution: In order to use D'Alembert's formula we need to extend initial conditions to all x, $\infty < x < \infty$. To this end, we first extend f with odd extension to $[-L, L]$ by letting

$$f_o(x) := \begin{cases} f(x) & \text{when } 0 \le x \le L, \\ -f(-x) & \text{when } -L \le x \le 0. \end{cases}$$

Notice that $f(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L})$ is an odd function therefore,

$$f_o(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L}) \quad \text{when } [-L, L].$$

Next step is to extend f_o to all $-\infty < x < \infty$ by extending periodically. That is,

$$F(x) = F(x + 2L) \quad \text{and} \quad F(x) = f_o(x) \text{ when } x \in [-L, L].$$

Notice that f_o is an periodic function with period $2L$. Hence

$$F(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L}) \quad \text{when } -\infty < x < \infty.$$

With the same approach, we first extend g with odd extension to $[-L, L]$ by letting

$$g_o(x) := \begin{cases} g(x) & \text{when } 0 \le x \le L, \\ -g(-x) & \text{when } -L \le x \le 0. \end{cases}$$

Notice that $g(x) = \frac{1}{2}\sin(\frac{2\pi x}{L})$ is an odd function therefore,

$$g_o(x) = \frac{1}{2}\sin(\frac{2\pi x}{L}) \quad \text{when } [-L, L].$$

Next step is to extend g_o to all $-\infty < x < \infty$ by extending periodically. That is,

$$G(x) = G(x + 2L) \quad \text{and} \quad G(x) = g_o(x) \text{ when } x \in [-L, L].$$

Notice that g_o is an periodic function with period $2L$. Hence

$$G(x) = \frac{1}{2}\sin(\frac{2\pi x}{L}) \quad \text{when } -\infty < x < \infty.$$

The solution we get from D'Alembert's formula is

$$u(x,t) = \frac{1}{2}[F(x+at) + F(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} G(r)dr = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr$$

where $f(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L})$ and $g(x) = \frac{1}{2}\sin(\frac{2\pi x}{L})$. Hence

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[f(x+at)+f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr \\
&= \frac{3}{2}\sin(\frac{\pi(x+at)}{L}) + \frac{3}{2}\sin(\frac{\pi(x-at)}{L}) - \frac{1}{2}\sin(\frac{4\pi(x+at)}{L}) - \frac{1}{2}\sin(\frac{4\pi(x-at)}{L})] + \frac{1}{4a}\int_{x-at}^{x+at}\sin(\frac{2\pi r}{L}) \\
&= 3\sin(1/2(\frac{\pi(x+at)}{L} + \frac{\pi(x-at)}{L}))\cos(1/2(\frac{\pi(x+at)}{L} - \frac{\pi(x-at)}{L})) \\
&- \sin(1/2(\frac{4\pi(x+at)}{L} + \frac{4\pi(x-at)}{L})\cos(1/2(\frac{4\pi(x+at)}{L} + \frac{4\pi(x-at)}{L})) \\
&- \frac{1}{4a}\frac{1}{\frac{2\pi r}{L}}\cos(r)|_{x-at}^{x+at} \\
&= 3\sin(\frac{\pi x}{L})\cos(\frac{\pi at}{L}) - \sin(\frac{4\pi x}{L})\cos(\frac{4\pi at}{L}) - \frac{L}{8\pi ra}[\cos(x+at) - \cos(x-at)] \\
&= 3\sin(\frac{\pi x}{L})\cos(\frac{\pi at}{L}) - \sin(\frac{4\pi x}{L})\cos(\frac{4\pi at}{L}) - \frac{L}{4\pi a}\sin(\frac{2\pi at}{L})\sin(\frac{2\pi x}{L})
\end{aligned}$$

which is exactly the same solution as in 1a in the previous problem.

Here we have used the following trig identities.

$$\sin(x) + \sin(y) = 2\sin((x+y)/2)\cos((x-y)/2)$$

and

$$\cos(x) - \cos(y) = -2\sin((x+y)/2)\sin((x-y)/2).$$

Question 7 (Exercise 5.3, 2) *Solve*

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le \pi,\ -\infty < t < \infty \\ u_x(0,t) = 0 \quad u_x(\pi,t) = 0 & -\infty < t < \infty, \\ u(x,0) = \cos^2(x) & 0 \le x \le \pi, \\ u_t(x,0) = \sin^2(x) & 0 \le x \le \pi. \end{cases}$$

1. *Using the Fourier series approach,*
2. *Using the method of images.*

Solution: Using the Fourier series approach. To do this, we first rewrite

$$\cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2} \quad \text{and} \quad \sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2}.$$

We know from (you do not need to know/memorize this) page 321 equation (4) that the general solution is

$$u(x,t) = A_0 t + B_0 + \sum_{n=1}^{\infty}[A_n \sin(\frac{n\pi at}{\pi}) + B_n\cos(\frac{n\pi at}{\pi})]\cos(\frac{n\pi x}{\pi})$$

Or after simplification,

$$u(x,t) = A_0 t + B_0 + \sum_{n=1}^{\infty}[A_n\sin(nat) + B_n\cos(nat)]\cos(nx)$$

Now using the initial condition, we will find A_n and B_n.

$$u(x,0) = \frac{1}{2} + \frac{\cos(2x)}{2} = 0 + B_0 + \sum_{n=1}^{\infty}[A_n\sin(0) + B_n\cos(0)]\cos(nx) = B_0 + \sum_{n=1}^{\infty} B_n\cos(nx)$$

From which we get $B_0 = 1/2$ and $B_2 = 1/2$ and all other B_n are zero. We next use second initial condition to figure out A_n.

$$u_t(x,0) = \frac{1}{2} - \frac{\cos(2x)}{2} = A_0 + \sum_{n=1}^{\infty}[A_n na\cos(0) - B_n na\sin(0)]\cos(nx) = A_0 + \sum_{n=1}^{\infty} A_n na\cos(nx).$$

From this we get that $A_0 = 1/2$ and $A_2 2a = -1/2$ and all other A_n is zero. Hence

$$u(x,t) = \frac{1}{2}t + \frac{1}{2} + \frac{1}{2}\cos(2at)\cos(2x) - \frac{1}{4a}\sin(2at)\cos(2x)$$

is the solution we are looking for.

Using the method of images. To this end, we need to extend the initial conditions from $0 \leq x \leq \pi$ to $-\infty < x < \infty$. In order the extension have desired boundary conditions we will use even extension. Let $f(x) = u(x,0) = \frac{1}{2} + \frac{\cos(2x)}{2}$ and $g(x) = u_t(x,0) = \frac{1}{2} - \frac{\cos(2x)}{2}$. Then

$$f_e(x) = \begin{cases} f(x) = \frac{1}{2} + \frac{\cos(2x)}{2} & \text{when } 0 \leq x \leq \pi, \\ f(-x) = \frac{1}{2} + \frac{\cos(-2x)}{2} & \text{when } 0 \leq x \leq \pi, \end{cases}$$

Notice that $f(x)$ is even function therefore we will have $f_e(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$ on $[-\pi, \pi]$. Similarly,

$$g_e(x) = \begin{cases} g(x) = \frac{1}{2} - \frac{\cos(2x)}{2} & \text{when } 0 \leq x \leq \pi, \\ g(-x) = \frac{1}{2} - \frac{\cos(-2x)}{2} & \text{when } 0 \leq x \leq \pi, \end{cases}$$

Similarly, $g(x)$ is even therefore, $g_e(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$ on $[-\pi, \pi]$. We next extend f_e and g_e to all $-\infty < x < \infty$ into periodic functions $F(x)$ and $G(x)$ with period of 2π

$$F(x+2\pi) = F(x) \quad F(x) = f_e(x) = \frac{1}{2} + \frac{\cos(2x)}{2} \text{ on } [-\pi, \pi].$$

and

$$G(x+2\pi) = G(x) \quad G(x) = g_e(x) = \frac{1}{2} - \frac{\cos(2x)}{2} \text{ on } [-\pi, \pi].$$

Notice that $f_e(x)$ is already periodic function with period of 2π (its 2π periodic extension will be itself). Hence $F(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$. Similarly, g_e is also periodic function with period 2π, hence $G(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$. Therefore, we want to solve

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty, \ -\infty < t < \infty \\ u(x,0) = F(x) = \frac{1}{2} + \frac{\cos(2x)}{2} & -\infty < x < \infty, \\ u_t(x,0) = G(x) = \frac{1}{2} - \frac{\cos(2x)}{2} & -\infty < x < \infty. \end{cases}$$

Notice that $F(x)$ is C^2 function as it is a trig function and $G(x)$ is C^1 function with similar reason we can use D'Alambert's formula to get

$$\begin{aligned} u(x,t) &= \frac{1}{2}[F(x-at) + F(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} G(r)dr \\ &= \frac{1}{2}[\frac{1}{2} + \frac{\cos(2(x-at))}{2} + \frac{1}{2} + \frac{\cos(2(x+at))}{2}] + \frac{1}{2a}\int_{x-at}^{x+at}[\frac{1}{2} - \frac{\cos(2r)}{2}]dr \\ &= \frac{1}{2} + \frac{\cos(2(x-at)) + \cos(2(x+at))}{4} + \frac{1}{2a}[\frac{r}{2} - \frac{\sin(2r)}{4}]_{x-at}^{x+at} \\ &= \frac{1}{2} + \frac{\cos(2(x-at)) + \cos(2(x+at))}{4} + \frac{1}{2a}[at - \frac{\sin(x+at) - \sin(x-at)}{4}] \end{aligned}$$

After a little algebra the solution we found with the fourier series method is exactly the same as with the solution we found with the method of images.

Question 8 (Exercise 5.3, 6) *Find the solution $u(x,t)$ to the following Wave equation*

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le \pi, \ -\infty < t < \infty, \\ u_x(0,t) = -1 \text{ and } u_x(L,t) = 1 & -\infty < t < \infty, \\ u(x,0) = \frac{x^2}{\pi} - x + 2\cos(3x) \text{ and } u_t(x,0) = \cos(x) & 0 \le x \le \pi. \end{cases}$$

Solution: We will follow the steps while converting the non-homogeneous boundary conditions to homogeneous boundary conditions for heat equation. Therefore, we shall find a particular solution $u_p(x,t)$ to Wave equation so that if we let

$$v(x,t) = u(x,t) - u_p(x,t)$$

then $v(x,t)$ solves the Wave equation with homogeneous boundary conditions. To this end, this particular solution given to us when $u_x(0,t) = c$ and $u_x(L,t) = d$

$$u_p(x,t) = \frac{a^2(d-c)}{2L} t^2 + \frac{d-c}{2L} x^2 + cx.$$

In our case $d = 1$ and $c = -1$ therefore,

$$u_p(x,t) = \frac{a^2(1-(-1))}{2\pi} t^2 + \frac{1-(-1)}{2\pi} x^2 + (-1)x = \frac{a^2 2}{2\pi} t^2 + \frac{2}{2\pi} x^2 - x.$$

Hence

$$v(x,t) = u(x,t) - u_p(x,t) = u(x,t) - [\frac{a^2}{\pi} t^2 + \frac{1}{\pi} x^2 - x].$$

Now

$$v_x(x,t) = u_x(x,t) - (u_p)_x)(x,t) = u_x(x,t) - (\frac{2x}{\pi} - 1).$$

From this we have

$$v_x(0,t) = u_x(0,t) - (u_p)_x)(0,t) = -1 - (-1) = 0$$

and

$$v_x(\pi,t) = u_x(\pi,t) - (u_p)_x)(\pi,t) = 1 - (2-1) = 0.$$

Hence v satisfies the homogeneous boundary conditions. We see that the choice of u_p is right. Moreover, $u_{tt} - a^2 u_{xx} = 0$ and $(u_p)_{tt} - a^2 (u_p)_{xx} = \frac{2a^2}{\pi} - a^2 \frac{2}{\pi} = 0$. Hence we get

$$v_{tt} - a^2 v_{tt} = u_{tt} - a^2 u_{xx} - ((u_p)_{tt} - a^2 (u_p)_{xx}) = 0.$$

Finally we check the initial conditions for v

$$v(x,0) = u(x,0) - u_p(x,0) = \frac{x^2}{\pi} - x + 2\cos(3x) - [\frac{1}{\pi} x^2 - x] = 2\cos(3x).$$

Similarly,

$$v_t(x,0) = u_t(x,0) - (u_p)_t(x,0) = \cos(x) - 0 = \cos(x).$$

Combining all of these we see that v solves the following Wave equation

$$\begin{cases} v_{tt} = a^2 v_{xx} & 0 \le x \le \pi, \ -\infty < t < \infty, \\ v_x(0,t) = 0 \text{ and } v_x(L,t) = 0 & -\infty < t < \infty, \\ v(x,0) = 2\cos(3x) \text{ and } v_t(x,0) = \cos(x) & 0 \le x \le \pi. \end{cases}$$

Now we can look for separable solution to this equation and observe that the general solution will be

$$v(x,t) = \sum_{n=1}^{\infty} [A_n \sin(\frac{n\pi at}{L}) + B_n \cos(\frac{n\pi at}{L})] \cos(\frac{n\pi x}{L}).$$

where $L = \pi$. So

$$v(x,t) = \sum_{n=1}^{\infty} [A_n \sin(nat) + B_n \cos(nat)] \cos(nx).$$

Now the difference here and the general solution we had in HW7 is that instead of sine we have cosine which is due to the boundary conditions are given in terms of x derivatives $v_x(0,t) = 0$ and $v_x(L,t) = 0$. When you run the machinery to find the solution that is what you will get for the solution. Now we use this general solution and the initial conditions to find A_n and B_n.

$$v(x,0) = 2\cos(3x) = \sum_{n=1}^{\infty} [A_n \sin(0) + B_n \cos(0)] \cos(nx) = \sum_{n=1}^{\infty} B_n \cos(nx)$$

From this we get $B_3 = 2$ and all other $B_n = 0$. Now for the second initial condition we need to find t derivative of the general solution,

$$v_t(x,t) = \sum_{n=1}^{\infty} [A_n na \cos(nat) - B_n an \sin(nat)] \cos(nx).$$

Evaluating this at $t = 0$ we get

$$v_t(x,0) = \cos(x) = \sum_{n=1}^{\infty} [A_n na \cos(0) - B_n an \sin(0)] \cos(nx) = \sum_{n=1}^{\infty} A_n na \cos(nx)$$

This gives us for $n = 1$, $A_1 a \cos(x) = \cos(x)$ or $A_1 = 1/a$ and all other $A_n = 0$. Combining all these we get (for B_3 and A_1)

$$v(x,t) = B_3 \cos(3at) \cos(3x) + A_1 \sin(at) \cos(x) = 2\cos(3at) \cos(3x) + \frac{1}{a} \sin(at) \cos(x).$$

We know that

$$v(x,t) = u(x,t) - u_p(x,t).$$

Hence

$$u(x,t) = v(x,t) + u_p(x,t) = 2\cos(3at) \cos(3x) + \frac{1}{a} \sin(at) \cos(x) + \frac{a^2}{\pi} t^2 + \frac{1}{\pi} x^2 - x$$

is the solution we are looking for.

Question 9 (Exercise 5.3, 7) *Find the solution $u(x,t)$ to the following Wave equation*

$$\begin{cases} u_{tt} = a^2 u_{xx} + e^{-t} \cos(x) & -\infty < x < \infty,\ -\infty < t < \infty, \\ u(x,0) = 0 \text{ and } u_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

Solution: We first observe that the Wave equation has a non-homogeneous right-hand side which is the function $h(t,x)$. Using Proposition 1 we have

$$u(x,t) = \frac{1}{2a} \int_0^t \int_{x-a(t-s)}^{x+a(t-s)} h(r,s) dr ds = \frac{1}{2a} \int_0^t \int_{x-a(t-s)}^{x+a(t-s)} e^{-s} cos(r) dr ds.$$

Hence we need to find this double integral

$$\begin{aligned}
u(x,t) &= \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} e^{-s} cos(r) dr ds \\
&= \frac{1}{2a}\int_0^t e^{-s}\sin(r)|_{r=x-a(t-s)}^{r=x+a(t-s)} \\
&= \frac{1}{2a}\int_0^t e^{-s}[\sin(x+a(t-s)) - \sin(x-a(t-s))]ds \\
&= \frac{1}{2a}\int_0^t e^{-s}\sin(x+a(t-s))ds - \frac{1}{2a}\int_0^t e^{-s}\sin(x-a(t-s))ds.
\end{aligned}$$

It remains to find the integral which I left it for you.

Question 10 (Exercise 5.3, 8) *Find the solution $u(x,t)$ to the following Wave equation*

$$\begin{cases} u_{tt} = a^2 u_{xx} + e^{-t}\cos(x) & -\infty < x < \infty,\ -\infty < t < \infty, \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

Here $f \in C^2$ and $g \in C^1$.

Solution: Now we are going to split the problem into two pieces. That is, let $u(x,t) = u_1(x,t) + u_2(x,t)$ where $u_1(x,t)$ solves the Wave equation with homogeneous right-hand side

$$\begin{cases} (u_1)_{tt} = a^2 (u_1)_{xx} & -\infty < x < \infty,\ -\infty < t < \infty, \\ u_1(x,0) = f(x) \text{ and } (u_1)_t(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

and $u_2(x,t)$ solves

$$\begin{cases} (u_2)_{tt} = a^2 (u_2)_{xx} + e^{-t}\cos(x) & -\infty < x < \infty,\ -\infty < t < \infty, \\ u_2(x,0) = 0 \text{ and } (u_2)_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

We focus on the first PDE here that u_1 solves. We know that the solution is given by the D'Alambert's formula ($f \in C^2$ and $g \in C^1$)

$$u_1(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr.$$

We now focus on the second pde u_2 solves. Notice that is exactly the same Wave equation with nonhomogeneous term on the right-hand side. Hence

$$u_2(x,t) = \frac{1}{2a}\int_0^t e^{-s}\sin(x+a(t-s))ds - \frac{1}{2a}\int_0^t e^{-s}\sin(x-a(t-s))ds.$$

Hence

$$\begin{aligned}
u(x,t) &= u_1(x,t) + u_2(x,t) \\
&= \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr \\
&+ \frac{1}{2a}\int_0^t e^{-s}\sin(x+a(t-s))ds - \frac{1}{2a}\int_0^t e^{-s}\sin(x-a(t-s))ds
\end{aligned}$$