UCONN
UNIVERSITY OF CONNECTICUT

Spring 2018 - Math 3435
Exam 1 - February 21
Time Limit: 50 Minutes

Name (Print): Solution KEY

This exam contains 6 pages (including this cover page) an empty scratch paper and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

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[1]Exam template credit: `http://www-math.mit.edu/~psh`

1. Consider the following two partial differential equations(PDEs)

$$(\star)\ \ L[u] = u_{xx} + u_{yy} = 0 \quad \text{and} \quad (\star\star)\ \ L[u] = u_t + 6uu_x - u_{xxx} = 0.$$

(a) (4 points) Find the order of the PDE

PDE in $(\star)$ has order 2 and PDE in $(\star\star)$ has order 3

(b) (3 points) Show that if the PDEs are **linear** or **non-linear** [Show your work]

PDE in $(\star)$ is Linear and PDE in $(\star\star)$ is non-linear

Solution: For PDE in $(\star)$ we see that if $u, v \in C^2$ functions and a is constant then

$$\begin{aligned} L[au+v] &= (au+v)_{xx} + (au+v)_{yy} = au_{xx} + v_{xx} + au_{yy} + v_{yy} \\ &= a(u_{xx} + u_{yy}) + v_{xx} + v_{yy} = aL[u] + L[v]. \end{aligned}$$

Hence the first PDE is a linear PDE.
For the PDE in $(\star\star)$ we see that if $u, v \in C^2$ functions and a is constant then

$$\begin{aligned} L[au+v] &= (au+v)_t + 6(au+v)(au+v)_x - (au+v)_{xxx} \\ &= au_t + v_t + 6a^2uu_x + 6auv_x + 6avu_x + 6vv_x - au_{xxx} - v_{xxx}. \end{aligned}$$

On the other hand,

$$aL[u] + L[v] = au_t + 6auu_x - au_{xxx} + v_t + 6vv_x - v_{xxx}.$$

Clearly $L[au+v] \neq aL[u] + L[v]$. Hence PDE in $(\star\star)$ is non-linear.

(c) (3 points) Consider $u_1(x,y) = x^3 + x^2 - y^2$ which solves $u_{xx} + u_{yy} = 6x$ and $u_2(x,y) = y^2 + 2x$ which solves $u_{xx} + u_{yy} = 2$. Can you find a function u which solves $u_{xx} + u_{yy} = 3x + 4$? [Show your work]. Solution: Notice that in part (b), it is shown that $L[u] = u_{xx} + u_{yy}$ is a linear PDE. We can use superposition principle find u which solves $u_{xx} + u_{yy} = 3x + 4$. Since $3x + 4 = \frac{1}{2}6x + 22$ we see that if we let $u = \frac{1}{2}u_1 + 2u_2$ then by superposition principle we have

$$L[u] = L[\frac{1}{2}u_1 + 2u_2] = \frac{1}{2}L[u_1] + 2L[u_2] = frac126x + 22 = 3x + 4.$$

From this we observe that $u = \frac{1}{2}u_1 + 2u_2$ solves $u_{xx} + u_{yy} = 3x + 4$. Since $u_1 = x^3 + x^2 - y^2$ and $u_2 = y^2 + 2x$ we get

$$u(x,y) = \frac{1}{2}u_1 + 2u_2 = \frac{1}{2}[x^3 + x^2 - y^2] + 2[y^2 + 2x].$$

is the solution we are looking for.

2. [Problems 1 and 3 from HW2]

(a) (6 points) Using ODE techniques find the general solutions of the following PDE for $u = u(x, y)$

$$u_x + 2xu = 4xy.$$

Solution: Here we should multiply the PDE with the integrating factor $\mu(x) = e^{\int 2x dx} = e^{x^2}$ to get

$$e^{x^2} u_x + e^{x^2} 2xu = e^{x^2} 4xy$$

Then the left hand side can be written as

$$(e^{x^2} u)_x = e^{x^2} 4xy.$$

Hence we now can integrate both sides with respect to x to get

$$\begin{aligned} e^{x^2} u(x,y) = \int (e^{x^2} u)_x dx &= \int e^{x^2} 4xy dx \\ &= 2y \int e^{x^2} 2x dx = 2y e^{x^2} + f(y) \end{aligned}$$

for some $f(y) \in C^1$. Hence general solution is

$$u(x,y) = 2y + \frac{f(y)}{e^{x^2}} \quad \text{for some } f \in C^1.$$

(b) (4 points) For the PDE in part (a), find a particular solution satisfying the side condition

$$u(x,x) = 0.$$

Solution: Since general solution is

$$u(x,y) = 2y + \frac{f(y)}{e^{x^2}} \quad \text{for some } f \in C^1.$$

We evaluate at $y = x$ to get

$$0 = u(x,x) = 2x + \frac{f(x)}{e^{x^2}}.$$

If we solve $f(x)$ to get

$$f(x) = -2xe^{x^2}.$$

Hence the particular solution is

$$u(x,y) = 2y + \frac{f(y)}{e^{x^2}} = 2y + \frac{-2ye^{y^2}}{e^{x^2}} = 2y(1 - e^{y^2 - x^2}).$$

3. (10 points) Solve the following first order constant coefficient PDE

$$4u_x - 3u_y = 0 \quad \text{subject to} \quad u(0,y) = y + y^3$$

Solution: We use the idea developed in section 2.1; and do the following change of variables; (notice that $a = 4$ and $b = -3$)

$$\begin{cases} w = -3x - 4y = constant & \text{this is same as } 3x + 4y = c\text{, up to you which one to use.} \\ z = y \end{cases}$$

We look for $v(w,z) = u(x,y)$ where w, z are the unknowns here. Since

$$u_x = v_w w_x + v_z z_x = v_w(-3) + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w(-4) + v_z$$

Using this if we rewrite our PDE in terms of v, w, z we get

$$0 = 4u_x - 3u_y = 4(-3)v_w - 3(-4v_w + v_z) = -12v_w + 12v_w - 3v_z.$$

Hence we have $v_z = 0$ which gives us $v(w,z) = f(w)$ for arbitrary $f \in C^1$. If we convert everything back to u, x, y to get

$$u(x,y) = v(w,z) = f(w) = f(-3x - 4y)$$

which is general solution we are looking for. We next use the given side condition to figure our f. Using general solution we found and given side condition we get

$$u(0,y) = f(0 - 4y) = y + y^3.$$

From this we get $f(-4y) = y + y^3$ or equivalently $f(t) = -t/4 + (-t/4)^3$. Hence the particular solution is

$$u(x,y) = f(-3x - 4y) = -\frac{-3x - 4y}{4} - \left(\frac{(-3x - 4y)}{4}\right)^3 = \frac{3x + 4y}{4} + \frac{(3x + 4y)^3}{4^3}.$$

4. [Problems 8 and 9 from HW3]

   (a) (6 points) Obtain the general solution of the following PDE

$$yu_x - 4xu_y = 2xy \quad \text{for all } (x, y).$$

Solution: In notation from section 2.2, we have $a(x,y) = y$ and $b(x,y) = -4x$. In order to make change of variables, we are looking for curves whose tangent at (x,y) is $b(x,y)/a(x,y)$ which will be parallel to $g(x,y) = a(x,y)\mathbf{i} + b(x,y)\mathbf{j}$. That is

$$\frac{dy}{dx} = \frac{b(x,y}{a(x,y)} = \frac{-4x}{y}.$$

We can solve this ordinary differential equation to find the curve we are looking for. Hence, we first get

$$ydy = -4xdx \quad \text{equivalently} \quad \frac{y^2}{2} = -2x^2 + c$$

That is, $\frac{y^2}{2} + 2x^2 = c$. Hence, we make the following change of variables;

$$\begin{cases} w = \frac{y^2}{2} + 2x^2 \\ z = y \end{cases}$$

We let $v(w,z) = u(x,y)$ and we now rewrite our PDE in terms of v and its derivatives in terms of w, z. To this end, we first compute

$$u_x = v_w w_x + v_z z_x = v_w 4x + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w y + v_z$$

Now we rewrite our PDE in terms of v and its derivatives with respect to w, z

$$2xy = yu_x - 4xu_y = y(4xv_w) - 4x(v_w y + v_z) = 4xyv_w - 4xyv_w - 4xv_z.$$

We have $2xy = -4xv_z$. From this we get $v_z = -y/2 = -z/2$ whenever $x \neq 0$. Integrating with respect to z in $v_z = -z/2$ we get

$$v(w,z) = -\frac{z^2}{4} + f(w) \quad \text{for arbitrary } f \in C^1.$$

Hence we get our solution $u(x,y)$ by converting everything back to u, x, y

$$u(x,y) = v(x,y) = -\frac{z^2}{4} + f(w) = -\frac{y^2}{4} + f(\frac{y^2}{2} + 2x^2) \quad \text{for arbitrary } f \in C^1.$$

   (b) (4 points) Find the particular solution of the PDE you found in (a) satisfying the side condition

$$u(x,0) = x^4.$$

Solution: As we get $u(x,y) = -\frac{y^2}{4} + f(\frac{y^2}{2} + 2x^2)$ we then use this and given side condition to find f;

$$x^4 = u(x,0) = 0 + f(0 + 2x^2).$$

From this we get $f(x) = x^2/4$. Hence

$$u(x,y) = -\frac{y^2}{4} + f(\frac{y^2}{2} + 2x^2) = -\frac{y^2}{4} + \frac{(y^2/2 + 2x^2)^2}{4}$$

5. Consider the following PDE

$$\begin{cases} u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0, & \text{The Heat equation} \\ u(0,t) = 0 \quad \text{and} \quad u(2,t) = 0, \quad t > 0, & \text{The boundary conditions} \\ u(x,0) = 3\sin(\pi x) - 4\sin(\frac{3\pi x}{2}), \quad 0 < x < 2 & \text{The initial condition.} \end{cases}$$

(a) (3 points) Which of the following solves the given heat equation with boundary conditions and initial condition [You should find the correct answer without solving the PDE].

1. $u(x,t) = 4e^{-\frac{2^2\pi^2}{2^2}t} - 3e^{-\frac{3^2\pi^2}{2^2}t}$.
2. $u(x,t) = 3\sin(\pi x) - 4\sin(\frac{3\pi x}{2})$.
3. $u(x,t) = 3\cos(\pi x)e^{-\frac{2^2\pi^2}{2^2}t} - 4\cos(\frac{3\pi x}{2})e^{-\frac{3^2\pi^2}{2^2}t}$.
4. $u(x,t) = 3\sin(\pi x)e^{-\frac{2^2\pi^2}{2^2}t} - 4\sin(\frac{3\pi x}{2})e^{-\frac{3^2\pi^2}{2^2}t}$.

Solution: It is easy to check that $u(x,0) = 3\sin(\pi x) - 4\sin(\frac{3\pi x}{2})$ matches only with functions in 2. and 4. It can be check that 2. does not satisfy the heat equation (notice that $u_t = 0$) we only left with 4. Hence the function in 4. is the solution to above heat equation.

(b) (3 points) Is the solution you found in part (a) the only solution? Can there be any other solutions to the above PDE?

Solution: Since the boundary conditions are 0 functions, i.e. C^2 and also the initial condition $3\sin(\pi x) - 4\sin(\frac{3\pi x}{2})$ is also C^2 function, from **the uniqueness theorem** there is at most one solution. As we have found a solution in the first part, it is the only solution. Hence there can not be any other solution.

(c) (4 points) Verify that the solution to above PDE satisfies $-7 \leq u(x,t) \leq 7$ for $0 \leq x \leq 2$ and $t \geq 0$.

Solution: Using the maximum principle, we get $u(x,t)$ is less than the maximum of boundary conditions and initial condition. As boundary conditions are zero, we only need to find the maximum of $3\sin(\pi x) - 4\sin(\frac{3\pi x}{2})$. Notice that sine is always bounded above by 1 we get $3\sin(\pi x) - 4\sin(\frac{3\pi x}{2}) \leq 3 + 4 = 7$. Hence $u(x,t) \leq \min(0,7) = 7$. Similarly, using the minimum principle we get $u(x,t) \geq$ minimum of 0 and $3\sin(\pi x) - 4\sin(\frac{3\pi x}{2})$ which is -7. Hence we get

$$-7 \leq u(x,t) \leq 7 \quad 0 \leq x \leq 3 \quad \text{and} \quad t \geq 0.$$

(d) [Bonus(5 points)] If we replace the initial condition with $u(x,0) = 0$, without solving the Heat equation, can you find the solution explicitly?

Solution: In this case,the maximum principle tells us that $u(x) \leq \max\{0,0,0\} = 0$ as boundary conditions are zero and the new initial condition is also zero. Hence $u(x,t) \leq 0$. Similarly minimum principle tells us that $0 \leq u(x,t)$. Now we have $0 \leq u(x,t) \leq 0$. If there is a solution to new PDE, then it has to be $u(x,t) = 0$. Notice that $u(x,t) = 0$ satisfies the heat equation, and the homogeneous boundary conditions as well as zero initial condition. Therefore $u(x,t) = 0$ is the solution in this case.