UCONN
UNIVERSITY OF CONNECTICUT

Spring 2018 - Math 3435
Exam 2 - March 28
Time Limit: 50 Minutes

Name (Print): Solution KEY

This exam contains 9 pages (including this cover page) an empty scratch paper and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	0	
Total:	40	

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[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. Let

$$f(x) = \begin{cases} 1 & \text{when } 0 \le x \le \pi, \\ 0 & \text{when } -\pi \le x < 0. \end{cases}$$

(a) (5 points) Find the Fourier series $\mathcal{F}(x)$ of $f(x)$ on $[-\pi, \pi]$

Solution: Notice that the function is neither even nor odd. Hence we have to find all the terms. We start with a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} 1dx = 1.$$

Then a_n, $n = 1, 2, \ldots,$

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\frac{n\pi x}{\pi})dx &= \frac{1}{\pi}\int_0^{\pi} 1\cos(\frac{n\pi x}{\pi})dx \\ &= \frac{1}{\pi}\frac{\sin(\frac{n\pi x}{\pi})}{\frac{n\pi}{\pi}}|_0^{\pi} \\ &= \frac{1}{\pi}\frac{\sin(n\pi)}{\frac{n\pi}{\pi}} - \frac{1}{\pi}\frac{1}{\frac{n\pi}{\pi}} \\ &= \frac{\sin(n\pi)}{n\pi} - \frac{0}{n\pi} = 0. \end{aligned}$$

We next find b_n, $n = 1, 2, \ldots,$

$$\begin{aligned} b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(\frac{n\pi x}{\pi})dx &= \frac{1}{\pi}\int_0^{\pi} 1\sin(\frac{n\pi x}{\pi})dx \\ &= \frac{1}{\pi}[-\frac{\cos(\frac{n\pi x}{\pi})}{\frac{n\pi}{\pi}}]|_0^{\pi} \\ &= -\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi} = -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi} \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{F}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(\frac{n\pi x}{\pi}) + b_n\sin(\frac{n\pi x}{\pi})] \\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n}\sin(\frac{n\pi x}{\pi}) \end{aligned}$$

(b) (2 points) At which points on $[-\pi, \pi]$, do $\mathcal{F}(x)$ and $f(x)$ **NOT** agree ?

Solution: Except at the end points and the discontinuity point, $\mathcal{F}(x) = f(x)$. Hence we only need to check points $x = -\pi, 0, \pi$. At 0, $f(0) = 1$ but $\mathcal{F}(1) = 1/2[1+0]/2$. Similarly, at $x = -\pi$, $f(x) = 0$ whereas $\mathcal{F}(-\pi) = 1/2 = [1+0]/2$. Finally, at $x = \pi$ we have $f(\pi) = 1$ and $\mathcal{F}(\pi) = 1/2 = [1+0]/2$. They do not agree at $x = -\pi, 0, \pi$ and at all other points they agree.

(c) (3 points) Verify that

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

Solution: Notice that $1-(-1)^n = 0$ when n is even and it is 2 when n is odd. Hence if we replace $n = 2k + 1$ we have

$$\mathcal{F}(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin(\frac{(2k+1)\pi x}{\pi}).$$

From part (b) we know that $\mathcal{F}(x) = f(x)$ at $x = \pi/2$. From this we get

$$1 = f(\frac{\pi}{2}) = \mathcal{F}(\frac{\pi}{2}) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)\frac{\pi}{2}).$$

After some algebra we get

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

2. (10 points) Solve

$$\begin{cases} u_t - u_{xx} = t\sin(2\pi x), & 0 \le x \le 1,\ t \ge 0, \\ u(0,t) = 0,\ \ u(1,t) = 0, & t \ge 1 \\ u(x,0) = \sin(3\pi x) & 0 \le x \le 1. \end{cases} \tag{1}$$

Solution: Since the boundary conditions are homogeneous, we can pass to the second step. That is we shall look for where $u(x,t) = u_1(x,t) + u_2(x,t)$ where u_1 solves the homogeneous heat equation;

$$\begin{cases} (u_1)_t - (u_1)_{xx} = 0, & 0 \le x \le 1,\ t \ge 0, \\ u_1(0,t) = 0,\ \ u_1(1,t) = 0, & t \ge 0, \\ u_1(x,0) = \sin(3\pi x) & 0 \le x \le 1. \end{cases} \tag{2}$$

and u_2 solves the non-homogeneous heat equation with zero initial condition

$$\begin{cases} (u_2)_t - (u_2)_{xx} = t\sin(2\pi x), & 0 \le x \le 1,\ t \ge 0, \\ u_2(0,t) = 0,\ \ u_2(1,t) = 0, & t \ge 0, \\ u_2(x,0) = 0 & 0 \le x \le 1. \end{cases} \tag{3}$$

Then by linearity of the heat equation we conclude that $u(x,t) = u_1(x,t) + u_2(x,t)$ solves our original equation (1). We shall first focus on u_1, we know the general solution is (you can use the proposition from the book, or our lecture notes)

$$u_1(x,t) = \sum_{n_1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

and using the initial condition for u_1 we get

$$u_1(x,0) = \sin(3\pi x) = \sum_{n_1}^{\infty} C_n \sin(n\pi x)$$

which tells us $C_3 = 1$ and all other $C_n = 0$. Hence

$$u_1(x,t) = e^{-9\pi^2 t} \sin(3\pi x).$$

solves (2). Now we focus on v_2. To solve (3), we shall use the Duhamel's principle. That is,

$$u_2(x,t) = \int_0^t \tilde{v}(x, t-s; s) ds$$

where $\tilde{v}$ solves

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = 0, & 0 \le x \le 1,\ t \ge 0, \\ \tilde{v}(0,t;s) = 0,\ \ \tilde{v}(1,t;s) = 0, & t \ge 0, \\ \tilde{v}(x,0;s) = s\sin(2\pi x) & 0 \le x \le 1. \end{cases} \tag{4}$$

Here you should think of s as a constant independent of t. We know that the general solution is

$$\tilde{v}(x,t;s) = \sum_{n_1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

and using the initial condition in (4) we get

$$\tilde{v}(x,0;s) = \sum_{n_1}^{\infty} C_n \sin(n\pi x) = s\sin(2\pi x)$$

which tells us that $C_2 = s$ and all other $C_n = 0$. Hence we have (for $n = 2$)

$$\tilde{v}(x,t;s) = se^{-4\pi^2 t}\sin(2\pi x).$$

Using Duhamel's principle we have

$$u_2(x,t) = \int_0^t \tilde{v}(x,t-s;s)ds = \int_0^t se^{-4\pi^2(t-s)}\sin(2\pi x)ds = \frac{1}{(4\pi^2)^2}(4\pi^2 t + e^{-4\pi^2 t} - 1)\sin(2\pi x).$$

Combining this with u_1 we get

$$u(x,t) = u_1(x,t) + u_2(x,t) = e^{-9\pi^2 t}\sin(3\pi x) + \frac{1}{(4\pi^2)^2}(4\pi^2 t + e^{-4\pi^2 t} - 1)\sin(2\pi x).$$

3. (10 points) Consider the Heat equation

$$\begin{cases} u_t - 4u_{xx} = \sin(2\pi x)t + 2xt, & 0 \le x \le 1,\ t \ge 0, \\ u(0,t) = \mathbf{1},\ \ u_x(1,t) = \mathbf{t^2}, \\ u(x,0) = 1 + \sin(3\pi x) - x. \end{cases} \tag{5}$$

where the boundary conditions are **non-homogeneous**. Transform the equation into a new one with **homogeneous** boundary conditions. (You do not need to solve the new equation).

Solution: We first should make the non-homogeneous boundary conditions homogeneous. To this end, we look for particular solution $u_p(x,t)$ to heat equation of the form $a(t)x + b(t)$. We first have $u(0,t) = b(t) = 1$. Also, $u_p(x,t) = a(t)x + 1$. Second boundary condition is given in terms x derivative, we first have $(u_p)_x(x,t) = a(t)$ and hence we should have $(u_p)_x(1,t) = a(t) = t^2$. Hence $u_p(x,t) = t^2x + 1$. We now let

$$v(x,t) = u(x,t) - u_p(x,t)$$

and hope that v will satisfy the heat equation with homogeneous boundary conditions.

$$v_t - 4v_{xx} = u_t - 4u_{xx} - [(u_p)_t - 4(u_p)_{xx}] = \sin(2\pi x)t + 2xt - [2tx - 0] = \sin(2\pi x)t.$$

Next, we check the boundary conditions

$$v(0,t) = u(0,t) - u_p(0,t) = 1 - 1 = 0 \quad \text{and} \quad v_x(1,t) = u_x(1,t) - (u_p(1,t))_x = t^2 - t^2 = 0.$$

Hence v satisfies the homogeneous boundary conditions. We next see the initial condition

$$v(x,0) = u(x,0) - u_p(x,0) = 1 + \sin(3\pi x) - x - 1 = \sin(3\pi x) - x$$

If we summarize what we got for v is that

$$\begin{cases} v_t - 4v_{xx} = \sin(2\pi x)t & 0 \le x \le 1,\ t \ge 0 \\ v(0,t) = 0,\ v_x(1,t) = 0 & t \ge 0, \\ v(x,0) = \sin(3\pi x) - x & 0 \le x \le 1. \end{cases}$$

4. (10 points) Describe the steps how to solve the following heat equation

$$\begin{cases} u_t - ku_{xx} = h(x,t), & 0 \le x \le \pi,\ t \ge 0, \\ u(0,t) = a(t),\ \ u(\pi,t) = b(t), & \\ u(x,0) = f(x). & \end{cases} \tag{6}$$

Solution:

Step 1: The boundary conditions are non-homogeneous, we will make them homogeneous. To do this, we let

$$u_p(x,t) := \frac{1}{L}(b(t) - a(t))x + a(t)$$

Now consider

$$v(x,t) = u(x,t) - u_p(x,t).$$

We should see that

$$v(0,t) = u(0,t) - u_p(0,t) = a(t) - a(t) = 0 \quad \text{and} \quad v(L,t) = u(L,t) - u_p(L,t) = b(t) - b(t) = 0.$$

On the other hand,

$$v_t - kv_{xx} = u_t - ku_{xx} - (u_p)_t + k(u_p)_{xx} = h(x,t) - \frac{1}{L}(b'(t) - a'(t))x - a'(t) =: H(x,t)$$

and

$$v(x,0) = u(x,0) - u_p(x,0) = f(x) - \frac{1}{L}(b(0) - a(0))x + a(0) = F(x)$$

Hence v satisfies the following equation

$$\begin{cases} v_t - kv_{xx} = H(x,t) & 0 \le x \le L,\ t \ge 0 \\ v(0,t) = 0 & v(L,t) = 0 \\ v(x,0) = F(x). & \end{cases}$$

Step 2: From this we consider $v(x,t) = v_1(x,t) + v_2(x,t)$ where where u_1 solves the homogeneous heat equation;

$$\begin{cases} (u_1)_t - (u_1)_{xx} = 0, & 0 \le x \le L,\ t \ge 0, \\ u_1(0,t) = 0,\ \ u_1(L,t) = 0, & \\ u_1(x,0) = F(x). & \end{cases} \tag{7}$$

and u_2 solves the non-homogeneous heat equation with zero initial condition

$$\begin{cases} (u_2)_t - (u_2)_{xx} = H(x,t), & 0 \le x \le L,\ t \ge 0, \\ u_2(0,t) = 0,\ \ u_2(L,t) = 0, & \\ u_2(x,0) = 0. & \end{cases} \tag{8}$$

Step 3: Find v_1. In case $F(x)$ is not given in terms of sine function we then need to do the half range extension and then find the Fourier series of $F(x)$ and finally find v_1.

Step 4: Using Duhamel's principle find v_2. That is,

$$u_2(x,t) = \int_0^t \tilde{v}(x, t-s; s) ds$$

where $\tilde{v}$ solves

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = 0, & 0 \le x \le L, \; t \ge 0, \\ \tilde{v}(0,t;s) = 0, \; \tilde{v}(L,t;s) = 0, & \\ \tilde{v}(x,0;s) = H(x,s;s). & \end{cases} \tag{9}$$

Step 5: Find $\tilde{v}$ first and in case $H(x,s;s)$ is not given in terms of sine function we then need to do the half range extension and then find the Fourier series of $H(x,s;s)$ and finally find $\tilde{v}$. Then find v_2.

Step 6: Combining all of these we get

$$u(x,t) = v(x,t) + u_p(x,t) = v_1(x,t) + v_2(x,t) + u_p(x,t).$$

5. (10 points (bonus)) For a given ϕ with $\lim_{x\to\pm\infty}\phi(x) = 0$ "rapidly", the solution to the following Heat conduction

$$\begin{cases} u_t = ku_{xx}, & -\infty \le x \le \infty,\ t \ge 0, \\ u(x,0) = \phi(x) & -\infty \le x \le \infty \end{cases} \tag{10}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} exp\left(-\frac{(y-x)^2}{4kt}\right) dx$$

Here $k > 0$.

(a) Using (10) show that $C(t) = \int_{-\infty}^{\infty} u(x,t)dx$ remains constant in time. You may assume $\lim_{x\to\pm\infty} u_x(x,t) = 0$. (Hint: Use (10) and do integration by parts to show that $C'(t) = 0$ for every $t \ge 0$).

Using the hint, we shall show that $C'(t) = 0$ for every $t \ge 0$.

$$C'(t) = \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_t(x,t)dx.$$

Using the PDE, we replace u_t in the integral by ku_{xx}

$$C'(t) = k\int_{-\infty}^{\infty} u_{xx}(x,t)dx = ku_x(x,t)|_{-\infty}^{\infty} = 0$$

as $\lim_{x\to\pm\infty} u_x(x,t) = 0$. Hence $C'(t) = 0$ we conclude that $C(t) =$constant.

(b) Using (10) show that $E(t) = \int_{-\infty}^{\infty} u^2(x,t)dx$ decreases in time. ((Hint: Use (10) and do integration by parts to show that $E'(t) < 0$ for every $t \ge 0$).

Solution: We once again use the hint here to show that $E'(t) < 0$ for every $t \ge 0$.

$$E'(t) = \frac{d}{dt}\int_{-\infty}^{\infty} u^2(x,t)dx = \int_{-\infty}^{\infty} 2u(x,t)u_t(x,t)dx.$$

Using the PDE, we replace u_t with ku_{xx} in the integral

$$E'(t) = k\int_{-\infty}^{\infty} 2u(x,t)u_{xx}(x,t)dx.$$

Now doing an integration by parts, and using boundary conditions, and $k > 0$, we have

$$\begin{aligned} E'(t) &= 2ku(x,t)u_x(x,t)|_{-\infty}^{\infty} - 2k\int_{-\infty}^{\infty} u_x(x,t)u_{xx}(x,t)dx \\ &= 0 - 0 - 2k\int_{-\infty}^{\infty} u_x^2(x,t)(x,t)dx \le 0 \end{aligned}$$

Hence we conclude that $E'(t) \le 0$ for every $t \ge 0$. This shows that E decreases in time.