UCONN
UNIVERSITY OF CONNECTICUT

Spring 2018 - Math 3435
Final Exam - April 30
Time Limit: 120 Minutes

Name (Print): Solution KEY

This solution key contains 15 pages (including this cover page) an empty scratch paper and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.
You may find the following identities useful in Question 5. The Laplace's equation in polar coordinates is

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0$$

and the general solution to this is

$$U(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

Also the following identity might be useful in the same question

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).$$

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	0	
Total:	80	

1

---

[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. (10 points) Use the method of characteristics the solve the first-order partial differential equation for $u = u(x,y)$

$$u_x - \frac{e^x}{1+e^y} u_y = 0 \quad \text{for } -\infty < x < \infty,\ y > 0$$

satisfying the side condition $u(x,0) = e^{2x}$.

Solution: In notation from section 2.2, we have $a(x,y) = 1$ and $b(x,y) = -\frac{e^x}{1+e^y}$. In order to make change of variables, we are looking for curves whose tangent at (x,y) is $b(x,y)/a(x,y)$ which will be parallel to $g(x,y) = a(x,y)\mathbf{i} + b(x,y)$. That is

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{-\frac{e^x}{1+e^y}}{1}.$$

We can solve this ordinary differential equation to find the curve we are looking for. Hence, we first get

$$(1+e^y)dy = -e^x dx \quad \text{equivalently} \quad y + e^y = -e^x + c$$

where we did integration to get this and c is arbitrary constant. From this we get

$$c = e^x + e^y + y.$$

Hence, we make the following change of variables;

$$\begin{cases} w = e^x + e^y + y, \\ z = y \end{cases}$$

We let $v(w,z) = u(x,y)$ and we now rewrite our PDE in terms of v and its derivatives in terms of w, z. To this end, we first compute

$$u_x = v_w w_x + v_z z_x = v_w e^x + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w(e^y + 1) + v_z$$

Then

$$0 = u_x - \frac{e^x}{1+e^y} u_y = v_w e^x - \frac{e^x}{1+e^y}[v_w(e^y+1) + v_z] = v_w e^x - v_w e^x - \frac{e^x}{1+e^y} v_z.$$

From this we get

$$0 = \frac{e^x}{1+e^y} v_z \quad \text{equivalently} \quad v_z = 0.$$

Since $v_z = 0$, integrating wrt z we get $v(w,z) = f(w)$. Since

$$u(x,y) = v(w,z) = f(w) = f(e^x + e^y + y).$$

We next use the prescribed condition to find f. Hence,

$$e^{2x} = u(x,0) = f(e^x + e^y + y) = f(e^x + 1).$$

If we let $t = e^x + 1$ then $e^x = t - 1$ and therefore, $f(t) = f(e^x + 1) = e^{2x} = (t-1)^2$. Finally,

$$u(x,y) = f(e^x + e^y + y) = (e^x + e^y + y - 1)^2.$$

2. Let $f(x)$ be given as

$$f(x) = x \quad \text{when} \quad 0 \le x < 1.$$

(a) (2 points) Extend $f(x)$ into an odd periodic function with period of 2.

Solution: As we want to extend f to be an odd function,

$$f_{\text{odd}}(x) = \begin{cases} f(x) = x & \text{when } 0 \le x < 1, \\ -f(-x) = x & \text{when } -1 < x \le 0. \end{cases}$$

Hence $f_{\text{odd}}(x) = x$ on $-1 < x < 1$. We let $f_{\text{odd}}(x) = f_{\text{odd}}(x+2)$ which is now odd function with period of 2.

(b) (4 points) Find Fourier series $\mathcal{F}(x)$ of the function you found in (a).

Solution: We want to find Fourier series $\mathcal{F}(x)$ of $f_{\text{odd}}(x)$. Since $f_{\text{odd}}(x)$ is an odd function we have coefficient of all cosine terms are zero,

$$a_0 = 0 \quad \text{and} \quad a_n = 0 \text{ for } n = 1, 2, \ldots.$$

Hence we only need to find coefficient of sine terms,

$$b_n = \frac{1}{1}\int_{-1}^{1} f_{\text{odd}}(x)\sin\left(\frac{n\pi x}{1}\right)dx = 2\int_0^1 f_{\text{odd}}(x)\sin(n\pi x)dx = 2\int_0^1 x\sin(n\pi x)dx.$$

If we integrate using the integration by parts we get $u = x$, $du = dx$ and $dv = \sin(n\pi x)dx$, $v = -\cos(n\pi x)/(n\pi)$. Hence

$$\begin{aligned} b_n &= 2\int_0^1 x\sin(n\pi x)dx = 2\left[-\frac{x\cos(n\pi x)}{n\pi}\Big|_{x=0}^{x=1} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx\right] \\ &= 2\left[-\frac{\cos(n\pi)}{n\pi} + 0 + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_{x=0}^{x=1}\right] \\ &= 2\left[-\cos(n\pi) + \frac{1}{(n\pi)^2}\sin(n\pi)\right] \\ &= -\frac{2}{n\pi}\cos(n\pi). \end{aligned}$$

Notice that $\cos(n\pi) = (-1)^n$ for $n = 1, 2, \ldots$. Hence

$$\begin{aligned} \mathcal{F}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + \sum_{n=1}^{\infty} b_n\sin(n\pi x) \\ &= 0 + 0 - \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin(n\pi x) \\ &= -\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin(n\pi x) \end{aligned}$$

(c) (4 points) Using part (a)-(b), verify that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Solution: Note that $f_{\text{odd}}(x)$ is continuous on $(-1,1)$ and therefore $\mathcal{F}(x) = f_{\text{odd}}(x)$ for every $x \in (-1,1)$. Now if we pick $x = 1/2$ then

$$\frac{1}{2} = f(\frac{1}{2}) = \mathcal{F}(\frac{1}{2}) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(\frac{n\pi}{2})$$

Note that $\sin(\frac{n\pi}{2}) = 0$ when n is even. Hence if we let $n = 2k+1$ for $k = 0, 1, \ldots$ then we have

$$\frac{1}{2} = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(\frac{n\pi}{2}) = -\sum_{k=0}^{\infty} \frac{2(-1)^{2k+1}}{(2k+1)\pi} \sin(\frac{(2k+1)\pi}{2})$$

Now $\sin(\frac{(2k+1)\pi}{2})$ is 1 when k is even and it is -1 when k is odd. Therefore $\sin(\frac{(2k+1)\pi}{2}) = (-1)^k$. First move $\pi/2$ to the left hand side, and use $(-1)^{2k+1} = -1$ and from these and above identity we get

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}$$

3. (10 points) Solve the Poisson equation

$$\begin{cases} u_{xx} + u_{yy} = x^2 + y^2, & x^2 + y^2 < 1, \\ u(x,y) = 0 & x^2 + y^2 = 1. \end{cases}$$

using the polar coordinates. You may want to look for solutions of the form $u(x,y) = f(x^2 + y^2)$.

Solution: As the hint is suggesting, we are going to look for solution $u(x,y) = f(x^2 + y^2)$. We first rewrite Laplace's equation in polar coordinates. We know that in polar coordinates

$$u_{xx} + u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = x^2 + y^2 = r^2$$

where $u(x,y) = U(r,\theta)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$ or $x^2 + y^2 = r^2$ and $\theta = \arctan(y/x)$. Now

$$U(r,\theta) = u(x,y) = f(x^2+y^2) = f(r^2) = F(r).$$

So we are going to look for solution $U(r,\theta)$ which only depends only on r. Therefore, $U_{\theta\theta} = 0$. We then have

$$r^2 = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = U_{rr} + \frac{1}{r}U_r$$

Say $W = U_r$ then above PDE becomes

$$r^2 = U_{rr} + \frac{1}{r}U_r = W_r + \frac{1}{r}W.$$

We can easily find integrating factor $\mu(r) = e^{\int 1/r dr} = r$. Multiply the above PDE with r to get

$$r^3 = rW_r + W = (rW)_r.$$

If we integrate both sides with respect to r we get

$$\frac{r^4}{4} + g(\theta) = rW$$

for some $g \in C^1$. Note that we are looking for solution depending only on r. Therefore, $g(\theta) = C$ for some C.

$$\frac{r^4}{4} + C = rW$$

Hence

$$W = \frac{r^3}{4} + \frac{C}{r}.$$

Remember that

$$U_r = W = \frac{r^3}{4} + \frac{C}{r}.$$

From this we get

$$U(r,\theta) = \frac{r^4}{16} + C\ln r + D$$

for some constants C, D. We now use the boundary condition $u(x,y) = 0$ when $x^2 + y^2 = 1$.

That is, $U(r,\theta) = 0$ when $r = 1$. Therefore,

$$0 = U(r = 1, \theta) = \frac{1^4}{16} + C \ln 1 + D = \frac{1}{16} + D.$$

From this we get $D = -1/16$ and hence

$$U(r,\theta) = \frac{r^4}{16} + C \ln r + D = \frac{r^4}{16} + C \ln r - \frac{1}{16}$$

for arbitrary C. We next observe that $\ln r$ blows up (goes to infinity when $r \to 0$) hence we choose $C = 0$ to get

$$U(r,\theta) = \frac{r^4}{16} - \frac{1}{16}.$$

We now convert everything back to Cartesian coordinate system

$$u(x,y) = U(r,\theta) = \frac{r^4}{16} - \frac{1}{16} = \frac{(x^2+y^2)^2}{16} - \frac{1}{16}.$$

4. (10 points) Suppose that $u(x,t)$ is solution of the diffusion equation with variable dissipation

$$\begin{cases} u_t - ku_{xx} + h(t)u = 0 & -\infty < x < \infty, \ t \geq 0, \\ u(x,0) = f(x) & -\infty < x < \infty. \end{cases}$$

and $g(t)$ is a solution to $g'(t) = h(t)g(t)$ with $g(0) = 1$. Then show that $v(x,t) = g(t)u(x,t)$ is a solution of

$$\begin{cases} v_t - kv_{xx} = 0 & -\infty < x < \infty, \ t \geq 0, \\ v(x,0) = f(x) & -\infty < x < \infty. \end{cases}$$

Solution: Since $v(x,t) = g(t)u(x,t)$ is given to us we then find $v_t - kv_{xx}$ first.

$$v_t = g'u + gu_t \quad \text{and} \quad v_{xx} = gu_{xx}.$$

Hence

$$v_t - kv_{xx} = g'u + gu_t - kgu_{xx}.$$

It is also given to us that $g(t)$ solves $g'(t) = h(t)g(t)$. Hence

$$v_t - kv_{xx} = g'u + gu_t - kgu_{xx} = hgu + gu_t - kgu_{xx} = hgu + g(u_t - ku_{xx}) = hgu + g(-hu) = 0$$

where we have also used that $u_t - ku_{xx} = -h(t)u$. Hence

$$v_t - kv_{xx} = 0.$$

we now check the boundary conditions;

$$v(x,0) = g(0)u(x,0) = 1f(x).$$

Combining all these we see that $v(x,t) = g(t)u(x,t)$ is a solution of

$$\begin{cases} v_t - kv_{xx} = 0 & -\infty < x < \infty, \ t \geq 0, \\ v(x,0) = f(x) & -\infty < x < \infty. \end{cases}$$

5. Let $u(x, y)$ be the solution to the following Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{when } x^2 + y^2 < 1, \\ u(x, y) = 4x^3 & \text{when } x^2 + y^2 = 1 \end{cases}$$

You may want to rewrite the Dirichlet problem in polar coordinates (including the boundary condition).

(a) (5 points) Find the solution $u(x, y)$. [See the cover page for the general solution and a hint].

Solution: As hint suggested we rewrite the Dirichlet in polar coordinates. The Laplace's equation in polar coordinates is

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0$$

and the boundary condition $u(x, y) = 4x^3$ should also be written in polar coordinates. As $x = r\cos(\theta)$ we get $4x^3 = r^3\cos^3(\theta)$. Using the other hint in the front page we have

$$4\cos^3(\theta) = 3\cos(\theta) + \cos(3\theta).$$

Hence we need to solve

and the general solution to this is

$$U(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

Also the following identity might be useful in the same question

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).$$

$$\begin{cases} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0 & \text{when } x^2 + y^2 = r^2 < 1, \\ U(r, \theta) = 3\cos(\theta) + \cos(3\theta) & \text{when } r^2 = x^2 + y^2 = 1. \end{cases}$$

It is also given that the general solution to Laplace equation in Polar coordinate is

$$U(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

When $r = 1$ we have

$$U(1, \theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)] = 3\cos(\theta) + \cos(3\theta).$$

From this we see that, $A_0 = 0$, all $B_n = 0$. Finally $n = 1$ we have $A_1\cos(\theta) = 3\cos(\theta)$. Hence $A_1 = 3$. For $n = 3$ we have $A_3\cos(3\theta) = \cos(\theta)$. Therefore, $A_3 = 1$. All $A_n = 0$. From this we get

$$U(r, \theta) = 3r\cos(\theta) + r^3\cos(3\theta).$$

(b) (2 points) Rewrite the solution you found in (a) in Cartesian coordinates, i.e. (x, y), and verify that it is the solution of the Laplace equation satisfying the given boundary condition.

Solution: We have $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

$$u(x,y) = U(r,\theta) = 3r\cos(\theta) + r^3\cos(3\theta) = 3x + r^3\cos(3\theta)$$

Now we need to find $r^3\cos(3\theta)$ in polar coordinates. One way to see this is

$$r^3\cos(3\theta) = \text{Re}(r^3 e^{3\mathbf{i}\theta}) = \text{Re}((re^{\theta})^3) = \text{Re}((x+\mathbf{i}y)^3)$$

Hence we need to find $\text{Re}((x+\mathbf{i}y)^3)$,

$$\text{Re}((x+\mathbf{i}y)^3) = \text{Re}(x^3 + 3x^2y - 3xy^2 - iy^3) = x^3 - 3xy^2.$$

From this we conclude that $u(x,y) = x^3 - 3xy^2 + 3x$. To verify that it satisfies the Laplace equation

$$u_{xx} = 6x \quad \text{and} \quad u_{yy} = -6x.$$

Hence $u_{xx} + u_{yy} = 6x - 6x = 0$. The boundary condition is that when $x^2 + y^2 = 1$, or $y^2 = 1 - x^2$. Then

$$u(x,y) = x^3 - 3xy^2 + 3x = x^3 - 3x(1-x^2) + 3x = x^3 - 3x + 3x^3 + 3x = 4x^3.$$

We see that $u(x,y) = x^3 - 3xy^2 + 3x$ satisfies the Laplace equation with boundary condition.

(c) (3 points) Find the maximum value of $u(x,y)$ in the disk of radius 1.

Since u is harmonic in the disk with radius 1 then by maximum principle we know that u attains its maximum on the boundary, i.e., circle centered at 0 with radius 1;

$$\max_{\bar{B}(0,1)} u(x,y) = \max_{\partial B(0,1)} u(x,y) = \max_{\partial B(0,1)} 4x^3 = 4.$$

6. (10 points) Find the solution $u(x,t)$ to the following Wave equation

$$\begin{cases} u_{tt} - a^2 u_{xx} = e^{-t}\cos(x) & -\infty < x < \infty,\ -\infty < t < \infty, \\ u(x,0) = \sin(x) \text{ and } u_t(x,0) = a\cos(x), & -\infty < x < \infty. \end{cases}$$

Solution: We first observe that the Wave equation has a non-homogeneous right-hand side which is the function $h(t,x)$. Notice that $h(x,t) = e^{-t}\cos(x)$ is smooth. Let $u = u_1 + u_2$ where u_1 solves

$$\begin{cases} (u_1)_{tt} - a^2 (u_2)_{xx} = 0 & -\infty < x < \infty,\ -\infty < t < \infty, \\ u_1(x,0) = \sin(x) \text{ and } (u_1)_t(x,0) = a\cos(x), & -\infty < x < \infty. \end{cases}$$

and u_2 solves

$$\begin{cases} (u_2)_{tt} - a^2 (u_2)_{xx} = e^{-t}\cos(x) & -\infty < x < \infty,\ -\infty < t < \infty, \\ u_2(x,0) = 0 \text{ and } (u_2)_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

Since Wave equation is linear and using the super position principle we know that $u = u_1 + u_2$ solves the original Wave equation above.

We focus on Wave equation u_1 solves. Since $f(x) = \sin(x)$ is at least C^2 function and $g(x) = a\cos(x)$ is at least C^1 we can use the D'Alambert's formula to get

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)dr = \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} a\cos(r)dr \\ &= \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{a}{2a}\sin(r)|_{x-at}^{x+at} \\ &= \frac{1}{2}[\sin(x+at) + \sin(x-at)] + \frac{1}{2}[\sin(x+at) - \sin(x-at)] \\ &= \sin(x+at). \end{aligned}$$

We now focus on the PDE that u_2 solves. As $h(x,t)$ is at least C^1 we can use Proposition 1 to have

$$u_2(x,t) = \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} h(r,s)drds = \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} e^{-s} cos(r)drds.$$

Hence we need to find this double integral

$$\begin{aligned} u_2(x,t) &= \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} e^{-s} cos(r)drds \\ &= \frac{1}{2a}\int_0^t e^{-s}\sin(r)|_{r=x-a(t-s)}^{r=x+a(t-s)} \\ &= \frac{1}{2a}\int_0^t e^{-s}[\sin(x+a(t-s)) - \sin(x-a(t-s))]ds \\ &= \frac{1}{2a}\int_0^t e^{-s}\sin(x+a(t-s))ds - \frac{1}{2a}\int_0^t e^{-s}\sin(x-a(t-s))ds. \end{aligned}$$

Hence

$$u(x,t) = u_1(x,t) + u_2(x,t) = \sin(x+at) + \frac{1}{2a}\int_0^t e^{-s}\sin(x+a(t-s))ds - \frac{1}{2a}\int_0^t e^{-s}\sin(x-a(t-s))ds.$$

7. Let $u(x,t)$ be the solution to the following heat equation

$$\begin{cases} u_t - u_{xx} = \frac{1}{\pi}xe^t + t[2 - \frac{2}{\pi}x + \sin(x)] & 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) = t^2,\ \ u(\pi,t) = e^t & t \ge 0, \\ u(x,0) = \frac{x}{\pi} + \sin(2x). \end{cases}$$

(a) (3 points) Find a particular solution u_p and let $v(x,t) = u(x,t) - u_p(x,t)$ so that $v(x,t)$ satisfies the homogeneous boundary condition and solves the non-homogeneous heat equation.
Solution: As the boundary conditions are non-homogeneous, the first step is to make them homogeneous. To this end, we let

$$u_p(x,t) = (b(t) - a(t))x/L + a(t) = (\frac{e^t - t^2}{\pi})x + t^2$$

so that $u_p(0,t) = t^2$ and $u_p(\pi,t) = e^t$. The second step is to let

$$v(x,t) = u(x,t) - u_p(x,t)$$

so that the non-homogeneous boundary conditions become homogeneous. Now v solves the non-homogeneous heat equation

$$v_t - v_{xx} = u_t - u_{xx} - [(u_p)_t - (u_p)_{xx}] = \frac{1}{\pi}xe^t + t[2 - \frac{2}{\pi}x + \sin(x)] - [(\frac{e^t - 2t}{\pi})x + 2t - 0] = t\sin(x).$$

The boundary conditions

$$v(0,t) = u(0,t) - u_p(0,t) = t^2 - t^2 = 0 \quad \text{and} \quad v(\pi,t) = u(\pi,t) - u_p(\pi,t) = e^t - e^t = 0.$$

The initial condition

$$v(x,0) = u(x,0) - u_p(x,0) = \frac{x}{\pi} + \sin(2x) - \frac{x}{\pi} = \sin(2x).$$

(b) (3 points) Write the PDE for which $v(x,t)$ solves, the boundary conditions and the initial condition $v(x,t)$ satisfies.
Solution: Hence, combining all of these we see that v solves

$$\begin{cases} v_t - v_{xx} = t\sin(x) & 0 \le x \le \pi,\ t \ge 0, \\ v(0,t) = 0 & v(\pi,t) = 0, \\ v(x,0) = \sin(2x). \end{cases} \tag{1}$$

(c) (4 points) Without solving the new equation corresponding to v describe how to solve it.
Solution: [You should describe the steps below without solving the problem] As v solves the non-homogeneous heat equation with initial conditions, the next step is to look for v_1, v_2 with $v(x,t) = v_1(x,t) + v_2(x,t)$ where v_1 solves homogeneous heat equation with

the initial condition in (1)

$$\begin{cases} (v_1)_t - (v_1)_{xx} = 0 & 0 \leq x \leq \pi,\ t \geq 0 \\ v_1(0,t) = 0 & v_1(\pi,t) = 0 \\ v_1(x,0) = \sin(2x). \end{cases} \tag{2}$$

and v_2 solves the non-homogeneous heat equation with zero initial condition

$$\begin{cases} (v_2)_t - (v_2)_{xx} = t\sin(x) & 0 \leq x \leq \pi,\ t \geq 0 \\ v_2(0,t) = 0 & v_2(\pi,t) = 0 \\ v_2(x,0) = 0. \end{cases} \tag{3}$$

We first focus on v_1. We know the general solution is

$$v_1(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx).$$

Using this and the given initial condition for v_1 we have

$$v_1(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) = \sin(2x)$$

which tells us that $C_2 = 1$ and all other $C_n = 0$. Hence we have

$$v_1(x,t) = e^{-4t}\sin(2x).$$

We now focus on v_2. From Duhamel's principle

$$v_2(x,t) = \int_0^t \tilde{v}(x, t-s; s) ds$$

where $\tilde{v}(x,t;s)$ solves the following homogeneous heat equation

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = 0 & 0 \leq x \leq \pi,\ t \geq 0 \\ \tilde{v}(0,t;s) = 0 & \tilde{v}(\pi,t;s) = 0 \\ \tilde{v}(x,0;s) = s\sin(x). \end{cases}$$

We know that the general solutions is

$$\tilde{v}(x,t;s) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx).$$

Using this and the initial condition for $\tilde{v}(x,t;s)$ we get

$$\tilde{v}(x,0;s) = s\sin(x) = \sum_{n=1}^{\infty} C_n \sin(nx)$$

From this, we see that $C_1 = s$ and all other $C_n = 0$. Hence

$$\tilde{v}(x,t;s) = se^{-t}\sin(x).$$

Using this we get

$$\begin{aligned} v_2(x,t) &= \int_0^t \tilde{v}(x,t-s;s)ds \\ &= \int_0^t se^{-(t-s)}\sin(x)ds \\ &= e^{-t}\sin(x)\int_0^t se^s ds \\ &= e^{-t}\sin(x)[te^t - e^t + 1] \\ &= \sin(x)[t - 1 + e^- t]. \end{aligned}$$

Hence

$$v(x,t) = v_1(x,t) + v_2(x,t) = e^{-4t}\sin(2x) + \sin(x)(t - 1 + e^- t).$$

Finally,

$$u(x,t) = v(x,t) + u_p(x,t) = e^{-4t}\sin(2x) + \sin(x)(t - 1 + e^{-t}) + (\frac{e^t - t^2}{\pi})x + t^2$$

is the solution we are looking for.

8. Assume that $u(x,t)$ satisfies the following diffusion equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & -\infty < x < \infty,\ t \geq 0, \\ u(x,0) = f(x),\ \ u_t(x,0) = g(x), & -\infty < x < \infty \end{cases}$$

where $f, g \in C^2$. Assume that $f(x)$ and $g(x)$ vanish when $|x|$ is big, say $|x| > 10^{3435}$. Define

$$F(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx \quad \text{and} \quad G(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx.$$

(a) (3 points) Show that $E(t) = F(t) + G(t)$ is constant for $t \geq 0$ (You may use $u_x(x,t) \to 0$ as $x \to \pm\infty$).

Solution: We show that $E'(t) = 0$ for every $t \geq 0$. This shows that $E(t)$ is constant for every $t \geq 0$.

$$\begin{aligned} E'(t) &= \frac{d}{dt}[\frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx] + \frac{d}{dt}[\frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx] \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}u_x^2(x,t)dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}u_t^2(x,t)dx \\ &= \int_{-\infty}^{\infty} u_x(x,t)u_{xt}(x,t)dx + \int_{-\infty}^{\infty} u_t(x,t)u_{tt}(x,t)dx \\ &= [u_x(x,t)u_t(x,t)]_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} u_{xx}(x,t)u_t(x,t)dx + \int_{-\infty}^{\infty} u_t(x,t)u_{tt}(x,t)dx \\ &= -\int_{-\infty}^{\infty} u_{xx}(x,t)u_t(x,t)dx + \int_{-\infty}^{\infty} u_t(x,t)u_{tt}(x,t)dx \\ &= -\int_{-\infty}^{\infty} u_{xx}(x,t)u_t(x,t)dx + \int_{-\infty}^{\infty} u_t(x,t)u_{xx}(x,t)dx = 0. \end{aligned}$$

(b) (3 points) Compute $E(0)$. (You may use $u_x(x,0) = f'(x)$.)

Solution: Direct computation gives us

$$\begin{aligned} E(0) = F(0) + G(0) &= \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,0)dx + \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,0)dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (f'(x))^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} g^2(x)dx \end{aligned}$$

(c) (4 points) Suppose that $E(t) = 0$ for every $t \geq 0$. Show that $u(x,t) =$ constant.

Solution: Since $E(t) = 0$ for every $t \geq 0$ we have

$$0 = E(t) = F(t) + G(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx + \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx.$$

From this we see that each $F(t) = 0$ and $G(t) = 0$. From these observations we have

$$0 = F(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx \quad \text{implies} \quad u_x(x,t) = 0.$$

Similarly,

$$0 = G(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx \quad \text{implies} \quad u_t(x,t) = 0.$$

Now we have $u(x,t)$ for which we see that $u(x,t) = \text{constant}$.

9. (10 points (bonus)) Use parts (a)-(c) in Question 8 to show that

$$\begin{cases} u_{tt} = u_{xx} & \text{when } -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) & \text{when } -\infty < x < \infty \end{cases}$$

has a unique solution. Here $f, g \in C^2$ and $f(x)$ and $g(x)$ vanish when $|x|$ is big, say $|x| > 10^{3435}$.

Solution: Suppose there are two solutions $u_1(x,t)$ and $u_2(x,t)$ solving the above Wave equation. Since Wave equation is linear if we let $v(x,t) = u_1(x,t) - u_2(x,t)$ then we see that v solves the Wave equation. Moreover, $v(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0$ and $v_t(x,0) = (u_1)_t(x,0) - (u_2)_t(x,0) = g(x) - g(x) = 0$. Therefore,

$$\begin{cases} v_{tt} - v_{xx} = 0 & -\infty < x < \infty,\ t \geq 0, \\ v(x,0) = 0 = f(x) \ \ v_t(x,0) = 0 = g(x), & -\infty < x < \infty \end{cases}$$

Now we know $E(t)$ is constant for every $t \geq 0$. That is $E(t) = E(0)$.

$$E(t) = F(t) + G(t) = \frac{1}{2}\int_{-\infty}^{\infty} v_x^2(x,t)dx + \frac{1}{2}\int_{-\infty}^{\infty} v_t^2(x,t)dx.$$

On the other hand, from part (b) we have (since $f = 0 = g$)

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty} (f'(x))^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} g^2(x)dx = 0$$

Hence $E(t) = E(0) = 0$. Now we can use part (c) to conclude that $v(x,t) =$ constant. Say $v(x,t) = c$. If we check the boundary condition, $v(x,0) = c = 0$ we get $v(x,t) = 0$ for every (x,t) with $-\infty < x < \infty$ and $t \geq 0$. In turn this gives, $u_1(x,t) - u_2(x,t) = v(x,t) = 0$. Hence $u_1(x,t) = u_2(x,t)$. Above pde has a unique solution.