UCONN
UNIVERSITY OF CONNECTICUT

Spring 2018 - Math 3435
Practice Exam 1 - February 21
Time Limit: 50 Minutes

Name (Print): ______________________________

This exam contains 8 pages (including this cover page) an empty scratch paper and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

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[1]Exam template credit: `http://www-math.mit.edu/~psh`

1. Consider the following two partial differential equations(PDEs)

$$(\star) \ \ L[u] = u_t - u_{xx} = 0 \quad \text{and} \quad (\star\star) \ \ L[u] = u - u_{xxt} + uu_{tt}.$$

(a) (4 points) Find the order of the PDE

PDE in $(\star)$ has order 2 and PDE in $(\star\star)$ has order 3

(b) (3 points) Show that if the PDEs are **linear** or **non-linear** [Show your work]

PDE in $(\star)$ is Linear and PDE in $(\star\star)$ is non-Linear.

Solution: Equation in $(\star)$ is linear as (a is arbitrary constant and u, v arbitrary C^2 functions)

$L[au+b] = (au+v)_t - (au+v)_{xx} = au_t + v_t - au_{xx} - v_{xx} = a(u_t - u_{xx}) + v_t - v_{xx} = aL[u] + L[v].$

On the other hand, equation in $(\star\star)$ is non-linear. Since (a, u, v are as above)

$$\begin{aligned} L[au+b] &= au + v - (au+v)_{xxt} + (au+v)(au+v)_{tt} \\ &= au + v - au_{xxt} - v_{xxt} + a^2uu_{tt} + auv_{tt} + avu_{tt} + vv_{tt}. \end{aligned}$$

On the other hand,

$$aL[u] + L[v] = a(u - u_{xxt} + uu_{tt}) + (v - v_{xxt} + vv_{tt}).$$

It is clear that $L[au+b] \neq aL[u] + L[v]$.

(c) (3 points) Suppose that u_1 solves $u_t - u_{xx} = f(x,t)$ and u_2 solves $u_t - u_{xx} = g(x,t)$ for some f, g. Can you find a function u which solves $u_t - u_{xx} = 3435f(x,t) + 2018g(x,t)$? [Show your work]. Solution: Notice that in part (b), it is shown that $u_t - u_{xx}$ is a Linear PDE. We can use **superposition principle** and to this end let $L[u] = u_t - u_{xx}$. Since $L[u_1] = f(x,t)$ and $L[u_2] = g(x,t)$, if we let $u = 3435u_1 + 2018u_2$ we have

$$L[u] = L[3435u_1 + 2018u_2] = 3435L[u_1] + 2018L[u_2] = 3435f(x,t) + 2018g(x,t).$$

From which we conclude that $u = 3435u_1 + 2018u_2$ solves $u_t - u_{xx} = 3435f(x,t) + 2018g(x,t)$.

2. (a) (6 points) Using ODE techniques find the general solutions of the following PDE for $u = u(x,y)$

$$yu_{xy} + 2u_x = x.$$

Solution: We now can rewrite the left-hand side of this PDE as $yu_{xy} + 2u_x = (yu_y + 2u)_x$. Therefore, if we integrate with respect to x first we get

$$yu_y + 2u = \int (yu_y + 2u)_x dx = \int x dx = \frac{x^2}{2} + f(y)$$

for some $f \in C^1$. Now we have $yu_y + 2u = x^2/2 + f(y)$ and the left-hand side has derivatives in terms of only y hence we think it kind of ODE and try to find integrating factor to do this we first divide all by y and get

$$u_y + 2/yu = \frac{x^2}{2y} + \frac{f(y)}{y}.$$

Now the integrating factor is $\mu(y) = e^{\int 2/y dy} = e^{\ln y^2} = y^2$. We multiply both sides of our PDE with this integrating factor to get

$$y^2u_y + 2y^2/yu = \frac{x^2y^2}{2y} + \frac{f(y)y^2}{y}.$$

After simplification we can rewrite our PDE as

$$(y^2u)_y = y^2u_y + 2y^2/yu = \frac{x^2y^2}{2y} + \frac{f(y)y^2}{y}.$$

Hence we can integrate both sides with respect to y to get

$$y^2u(x,y) = \int (y^2u)_y dy = \int (\frac{x^2y}{2} + yf(y))dy = \frac{x^2y^2}{4} + F(y) + G(x)$$

where we let $\int yf(y) = F(y)$ for arbitrary $F \in C^2$ and arbitrary $G \in C^2$. After some algebra we get general solution

$$u(x,y) = \frac{x^2}{4} + \frac{F(y)}{y^2} + \frac{G(x)}{y^2} \quad \text{for arbitrary } F, G \in C^2.$$

(b) (4 points) For the PDE in part (a), find a particular solution satisfying the side conditions

$$u(x,1) = 0 \quad \text{and} \quad u(0,y) = 0.$$

Solution: As we have side, we will use them to find F, G. To this end, if we use the first condition and solution we found in the first part we get

$$0 = u(x,1) = \frac{x^2}{4} + \frac{F(1)}{1^2} + \frac{G(x)}{1^2} = \frac{x^2}{4} + F(1) + G(x).$$

From this we get $G(x) = -\frac{x^2}{4} - F(1)$. On the other hand, If we use the second side condition we have

$$0 = u(0,y) = 0 + \frac{F(y)}{y^2} + \frac{G(0)}{y^2}.$$

From this we get $F(y) = G(0)$ as G is a function then $G(0)$ is some number therefore $F(y) = \text{constant} = c$. Hence combining all of these we have

$$u(x,y) = \frac{x^2}{4} + \frac{c}{y^2} + \frac{1}{y^2}\left(-\frac{x^2}{4} - c\right) = \frac{x^2}{4} - \frac{x^2}{4y^2}.$$

3. (10 points) Solve the following PDE

$$u_x - 2u_y = 0 \quad \text{subject to} \quad u(x, e^x) = e^{2x} + 4xe^x + 4x^2$$

Solution: We use the idea developed in section 2.1; and do the following change of variables; (notice that $a = 1$ and $b = -2$)

$$\begin{cases} w = -2x - y \\ z = y \end{cases}$$

We look for $v(w,z) = u(x,y)$ where w, z are the unknowns here. Since

$$u_x = v_w w_x + v_z z_x = v_w(-2) + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w(-1) + v_z$$

Using this if we rewrite our PDE in terms of v, w, z we get

$$0 = u_x - 2u_y = -2v_w - 2(-v_w + v_z) = -2v_w + 2v_w - 2v_z.$$

Hence we have $v_z = 0$ which gives us $v(w,z) = f(w)$ for arbitrary $f \in C^1$. If we convert everything back to u, x, y to get

$$u(x,y) = v(w,z) = f(w) = f(-2x - y)$$

which is general solution we are looking for. We next use the given side condition to figure our f. Using general solution we found and given side condition we get

$$u(x, e^x) = f(-2x - e^x) = e^{2x} + 4xe^x + 4x^2 = (e^x + 2x)^2.$$

From this we get $f(x) = x^2$. Hence the particular solution is

$$u(x,y) = f(-2x - y) = (-2x - y)^2.$$

4. (a) (6 points) Find the general solution of the following PDE

$$xu_x - xyu_y = 0 \quad \text{for all } (x, y).$$

Solution: In notation from section 2.2, we have $a(x,y) = x$ and $b(x,y) = -xy$. In order to make change of variables, we are looking for curves whose tangent at (x,y) is $b(x,y)/a(x,y)$ which will be parallel to $g(x,y) = a(x,y)\mathbf{i} + b(x,y)$. That is

$$\frac{dy}{dx} = \frac{b(x,y}{a(x,y)} = \frac{-xy}{x} = -y.$$

We can solve this ordinary differential equation to find the curve we are looking for. Hence, we first get

$$\frac{dy}{y} = -dx \quad \text{equivalently} \quad \ln|y| = -x + c$$

where we did integration to get this and c is arbitrary constant. From this we get $y = e^{-x}e^{c} = e^{-x}c$. From this we get $ye^{x} = c$. Hence, we make the following change of variables;

$$\begin{cases} w = ye^{x} \\ z = y \end{cases}$$

We let $v(w,z) = u(x,y)$ and we now rewrite our PDE in terms of v and its derivatives in terms of w, z. To this end, we first compute

$$u_x = v_w w_x + v_z z_x = v_w(ye^{x}) + 0 \quad \text{and} \quad u_y = v_w w_y + v_z z_x = v_w e^{x} + v_z$$

Now if rewrite our PDE in terms of v, w, z we get

$$xu_x - xyu_y = xv_w(ye^{x}) - xy(v_w e^{x} + v_z) = xye^{x}v_w - xye^{x}v_w - xyv_z = 0.$$

From this we see that $v_z = 0$. If we solve this we get $v(w,z) = f(w)$ for arbitrary $f \in C^1$. If we convert everything back to u, x, y we have

$$u(x,y) = v(w,z) = f(w) = f(ye^{x}).$$

is the general solution we are looking for.

(b) (4 points) Find the particular solution of the PDE you found in (a) satisfying the side condition

$$u(x,x) = x^2e^{2x}.$$

Solution: Now we figure out f using the given side condition

$$x^2e^{2x} = u(x,x) = f(xe^{x})$$

From this we get $f(x) = x^2$. Hence the particular solution is

$$u(x,y) = f(ye^{x}) = (ye^{x})^2.$$

5. Consider the following PDE

$$\begin{cases} u_t - 9u_{xx} = 0 \quad \text{whenever } 0 \le x \le 3,\ t \ge 0, \\ u(0,t) = 0 \quad \text{and} \quad u(3,t) = 0, \\ u(x,0) = 1417\sin(\pi x) + 2018\sin(3\pi x). \end{cases}$$

(a) (3 points) Which of the following solves the given heat equation.

1. $u(x,t) = 2018e^{-81t}\sin(\pi x) + 1417e^{-9^3 t}\sin(3\pi x)$.
2. $u(x,t) = 1417\sin(\pi x) + 2018\sin(3\pi x)$.
3. $u(x,t) = 1417e^{-81t}\sin(\pi x) + 2018e^{-9^3 t}\sin(3\pi x)$.
4. $u(x,t) = 1417e^{-81t}\cos(\pi x) + 2018e^{-9^3 t}\cos(3\pi x)$.

Solution: Here we should use the given conditions, specifically the last one, the initial condition to check. As

$$u(x,0) = 1417\sin(\pi x) + 2018\sin(3\pi x)$$

We see that only 2. and 3. satisfy this. It can be check that 2. does not satisfy the heat equation (notice that $u_t = 0$) we only left with 3. Hence

$$u(x,t) = 1417e^{-81t}\sin(\pi x) + 2018e^{-9^3 t}\sin(3\pi x)$$

is the solution to the heat equation.

(b) (3 points) Is the solution you found in part (a) the only solution? Can there be any other solutions to the above PDE? Solution: Since the boundary conditions are 0 functions, i.e. C^2 and also the initial condition $1417\sin(\pi x) + 2018\sin(3\pi x)$ is also C^2 function, from **the uniqueness theorem** there is at most one solution. As we have found a solution in the first part, it is the only solution. Hence there can not be any other solution.

(c) (4 points) Using the maximum and the minimum principles verify that the solution you found in (a) satisfies

$$-3435 \le u(x,t) \le 3435 \quad 0 \le x \le 3 \quad \text{and} \quad t \ge 0.$$

Solution: Using the maximum principle, we get $u(x,t)$ is less than the maximum of boundary conditions and initial condition. As boundary conditions are zero, we only need to find the maximum of $1417\sin(\pi x) + 2018\sin(3\pi x)$. Notice that sine is always bounded above by 1 we get $1417\sin(\pi x) + 2018\sin(3\pi x) \le 1417 + 2018 = 3435$. Similarly, using the minimum principle we get $u(x,t) \ge$ minimum of $1417\sin(\pi x) + 2018\sin(3\pi x)$ which is -3435. Hence we get

$$-3435 \le u(x,t) \le 3435 \quad 0 \le x \le 3 \quad \text{and} \quad t \ge 0.$$