BOUSSINESQ'S APPROXIMATION

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WHERE DOES IT COME FROM?

- The Navier-Stokes Equation (NSE): The general expression for conservation of momentum of an incompressible, Newtonian fluid in the NSE
  - Used to solve non-isothermal flow, such as natural convection problems, without having to solve for the full compressible formulation of the Navier-Stokes equations
- Boussinesq's Approximation (BA) is a derivation of the NSE which states that density is negligible in the NSE except in terms * the gravitational acceleration (g)
- Joseph Valentin Boussinesq 1872
  - First derived them in response to the observation by John Scott Russell of the wave of translation
  - Models water dispersion and widely used in science and engineering
    - Buoyancy
    - Water waves
    - Turbulence modeling and eddy viscosity

    - A means of studying the dynamics of fluids without necessarily having to resolve sound waves (filtered out)
      - Valid when domain considered is much shorter than a density or temperature scaleheight (which is smaller)
      - BTEs for liquids and gases are essentially similar, except for the thermal energy equation

DERIVATION

- The Navier-Stokes equations (NSEs) govern the motion of fluids in the general case of compressible fluids yielding

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \nabla\cdot\left(\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I}\right) + \rho\mathbf{g}$$

  - $\mathbf{u}$ fluid velocity
  - p fluid pressure
  - ρ fluid density
  - μ fluid dynamic viscosity
  - $\mathbf{I}$ identity matrix
  - $\mathbf{g}$ the acceleration due to gravity

$$g' = g\frac{\rho_1 - \rho_2}{\rho}.$$

- NSEs are solved with the continuity equation: $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\mathbf{u} = 0$
- The Boussinesq's approximation states that the density variation only pertains to buoyancy term pg, yielding

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \nabla\cdot\left(\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I}\right) + \rho\mathbf{g}$$

Note: Temperature and pressure-dependent density, ρ, have been replaced by a constant density, ρ_0, except in the body force term representing the buoyancy term

$\mathbf{u} \cdot \nabla \mathbf{u}$ =0

The general expression for conservation of momentum of an incompressible, Newtonian fluid (the Navier–Stokes equations) is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho}\mathbf{F},$$

If $\mathbf{F} = \rho g$ is the gravitational body force, the resulting conservation equation is[1]:129

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} - \mathbf{g}\alpha \Delta T.$$

In the equation for heat flow in a temperature gradient, the heat capacity per unit volume, ρC_p, is assumed constant. The resulting equation is

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + \frac{J}{\rho C_p},$$

where J is the rate per unit volume of internal heat production and κ is the thermal diffusivity.[1]:129

The three numbered equations are the basic convection equations in the Boussinesq approximation.

ISOTHERMAL VS. NONISOTHERMAL FLOW

Isothermal

- Def: having a constant temperature
- Engineering application: a fluid's temperature will remain constant because variations are either very small or inconsequential in magnitude compared to other physical variations involved

Nonisothermal

- Refers to fluid flows with temperatures that are not constant
- Fluids subject to temperature changes experience material property changes as well (density and viscosity)

INCOMPRESSIBILITY CONDITION

- Restrict analysis to that of systems whose background density and temperature do not vary much overall around their mean values
- Ocean
- Earth's atmosphere
- Stellar interiors

USES AND ASSUMPTIONS

- Used to solve nonisothermal flow equations because the computational costs are lower and convergence was more likely to be achieved
- Accurate approximation when density variations are small
  - Slightly reduces nonlinearity of the problem by eliminating extra non-constant variables
  - Assumes variations in density have no effect on the flow field, except that they give rise to buoyancy forces (terms * g)

$$g' = g\frac{\rho_1 - \rho_2}{\rho}.$$

- Sound waves impossible/neglected (move via density variations)
- Can be used for fluids moving in different directions but best when moving in one direction
- Used to model liquids around room temperature, natural ventilation in buildings, or dense gas dispersion in industrial set-ups
  - Example:
    - Open window in a house letting in fresh air

      - Applied to problems where the fluid varies in temperature from one place to another, driving flow of fluid and heat transfer. Variations in fluid properties, other than density, are ignored.

REAL WORLD APPLICATIONS

- Slight reduction of nonlinearity of the system + today's computational hardware = less prevalent + marginal reduction in computational costs (cc)
- Navier-Stokes equation cc < Boussinesq's Equation cc
- Water waves
  - An approximation valid for weakly non-linear and fairly long waves
  - Takes into account the vertical structure of the horizontal and vertical flow velocity = non-linear partial differential equations called Boussinesq-type Equations (BTE)
- In coastal engineering, BAs are frequently used in computer models for the simulation of water waves in shallow seas and harbors, but best for fairly long waves
  - Stoke's expansion is more appropriate for short waves (when the wavelength is of the same order as the water depth, or shorter)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho} \mathbf{F},$$

REAL WORLD APPLICATIONS

- For the simulation of wave motion near coasts and harbors, numerical models-both commercial and academic- BTEs exist
- Commercial Examples:
  - BT Wave Modules in MIKE 21 and SMS
    - MIKE 21: a computer program that simulates flows, waves, sediments and ecology in rivers, lakes, estuaries, bays, coastal areas and seas in two dimensions
      - Developed by DHI
      - BTEs enable you to model
        - Operational and design conditions within ports and harbors
        - Wave disturbance inside harbor basins
        - Surf and swash zone dynamics
        - Propagation and transformation of ship-generated waves and tsunamis in harbors and coastal areas
    - SMS (Hydrology Software): a complete program for building and simulating surface water models
      - Features 1D and 2D modeling and a unique conceptual model approach
        - Wave model of nearshore waves travelling towards a harbor entrance
  - Free Boussinesq Models are Celeris, COULWAVE, and FUNWAVE
    - Celeris: shows wave breaking and refraction near beaches
      - Interactive model

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