HUNTER-SAXTON EQUATION

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Introduction

- Equation
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Equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2$$

Hunter and Saxton

- Dr. John Hunter: Mathematics Professor at University of California, Davis
- Dr. Ralph Saxton: Mathematics Professor at University of New Orleans

Nematic Liquid Crystals

- Nematic = “thread” of disclinations
  - *No individual positional order, but overall directional order*
    - Free to flow
  - *Orient over one major axis*
- Liquid crystal structure aligned -> slightly wiggled
  - *Orientation Waves*

Why does this equation matter? To most people, it really doesn’t, but to people working on the theoretical study of nematic liquid crystals this integrable PDE is an important tool.
Nematic means thread. Used to refer to the crystals because throughout the crystal structure, there is a thread of disclinations, or line defects in which rotational symmetry is violated. The calamitic (rod-shaped organic molecules pictured) have no individual order, but there is an overall directional order, meaning there is a semi-structure when looked at as a whole but each individual molecule is free to flow making their center of mass positions random relative to one another. This quality is the reason that the Nematic Crystals are referred to as liquid crystals. This overall order is centered around one major axis. The Hunter-Saxton equation comes into play regarding the “thread of defects” concept. A disturbance can cause aligned calamitics to wiggle which can then propagate through the entire crystal structure. These disturbances are called orientation waves and can be described with the Hunter-Saxton equation.

From Theory to Practice

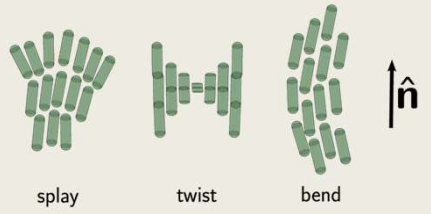
The fluid nature of Nematic Liquid Crystals makes the structure easily aligned by external magnetic or electric fields, making them ideal for use in liquid-crystal displays (LCDs). An LCD is an “electronically modulated optical device that uses the light-modulating properties of liquid crystals”, thus knowing and controlling how the liquid crystal structure is aligned is extremely helpful in real life applications.

Director Field

- Orientation of molecules
  - *Described by **director field** (based on the elastic continuum theory)*

$$\mathbf{n}(x, y, z, t) = (\cos\varphi(x, t), \sin\varphi(x, t), 0)$$

$$W(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}\left(\alpha(\nabla \cdot \mathbf{n})^2 + \beta(\mathbf{n} \cdot (\nabla \times \mathbf{n}))^2 + \gamma|\mathbf{n} \times (\nabla \times \mathbf{n})|^2\right)$$



Fair warning: the derivation of this equation, which corresponds to what the terms in the PDE mean, is based on other principles, equations, and theories that are way beyond the scope of this class, so I might sight some of them without really explaining why you can. The Hunter-Saxton equation only considers the orientation of the molecules. This orientation is described by a field of unit vectors called the director field due to the elastic continuum theory. This theory considers the crystal a continuous sequence, so specific molecular details are ignored and the distortion to the oriented structure are considered. The potential energy density of this field is described by this equation take from the Oseen-Frank energy functional, where alpha corresponds to splay, beta corresponds to twist, and gamma corresponds to bend; all of these are elastic coefficients. The director field listed above is based on the fact that Hunter and Saxton only considered “splay waves”.

Derivation

$$\mathcal{L} = \frac{1}{2}\left(\varphi_t^2 - a^2(\varphi)\varphi_x^2\right), \qquad a(\varphi) := \sqrt{\alpha \sin^2 \varphi + \gamma \cos^2 \varphi}$$

$$\varphi_{tt} = a(\varphi)[a(\varphi)\varphi_x]_x$$

$$\varphi(x,t;\epsilon) = \varphi_0 + \epsilon\varphi_1(\theta,\tau) + O(\epsilon^2)$$

$$(\varphi_{1\tau} + a'(\varphi_0)\varphi_1\varphi_{1\theta})_\theta = \frac{1}{2}a'(\varphi_0)\varphi_{1\theta}^2$$

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2$$

Taking the equation from the previous slide, the splay-assumption that restricts the director field, and the case of the Lagrangian (which will be taken as a given), the combination of the potential energy from the previous slide and kinetic energy considerations is the first equation. With this assumption, the Euler-Lagrange equation (which is another partial differential equation for which the solutions are the functions for which a given functional is stationary) for the angle phi is the second. Linearization of this equation around the equilibrium phi-not (angle where the molecules are perfectly aligned, with a-not being the speed at the angle phi-not). The non-linear equation can be expected to behave in a similar manner. Equation 3 represents propagation waves with a large t. When its asymptotic solutions are inserted into the equation, the forth equation is generated, which can be rewritten as the Hunter-Saxton equation.

Unknown and Terms

- The unknown *u*
  - *Corresponds to "splay waves"*
  - *In terms of φ (alignment angle)*
- φ changes with time and distance
  - *T and X derivatives describe these changes*

From the derivation, the unknown u corresponds to "splay waves" that propagate throughout the liquid crystal molecule. It is in terms of the alignment angle phi which changes with time and distance. Thus, the t and x derivatives in the Hunter-Saxton equation models these changes. As stated in the nematic liquid crystals slide, the Hunter-Saxton equation describes the orientation waves that cause disclinations throughout the crystal's structure-how the alignment angle changes over time and distance directly links to this concept.

Not Just Splaying?

$$\mathbf{n}(x,y,z,t) = (\cos\varphi(x,t), \sin\varphi(x,t)\cos\psi(x,t), \sin\varphi(x,t)\sin\psi(x,t))$$

$$\mathcal{L} = \frac{1}{2}\left(\varphi_t^2 - a^2(\varphi)\varphi_x^2 + \sin^2\varphi\left[\psi_t^2 - b^2(\varphi)\psi_x^2\right]\right), \qquad a(\varphi) := \sqrt{\alpha\sin^2\varphi + \gamma\cos^2\varphi}, \quad b(\varphi) := \sqrt{\beta\sin^2\varphi + \gamma\cos^2\varphi}$$

$$(v_t + uv_x)_x = 0, \qquad u_{xx} = v_x^2$$

$$\left[(u_t + uu_x)_x - \frac{1}{2}u_x^2\right]_x = 0$$

The more general version the director field does not restrict the motion to solely splay waves. When the possibility of twist waves is considered, both phi and psi (the twist angle) can change, which results in the Lagrangian of the second equation (a slightly more complicated version of the derivation before). The asymptotic analysis of this equation results in the third block of equations where u is in terms of phi and v is in terms of psi to represent both the splaying and twisting. However, this system still implies that u satisfies the fourth equation, and thus the Hunter-Saxton equation still holds.

References

- Euler–Lagrange equation. (2018, February 28). Retrieved March 06, 2018, from https://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation
- Hunter, J. K. (n.d.). John K. Hunter. Retrieved March 24, 2018, from https://www.math.ucdavis.edu/~hunter/
- Hunter–Saxton equation. (2018, February 28). Retrieved March 06, 2018, from https://en.wikipedia.org/wiki/Hunter%E2%80%93Saxton_equation
- Liquid-crystal display. (2018, March 03). Retrieved March 06, 2018, from https://en.wikipedia.org/wiki/Liquid-crystal_display
- Liquid crystal. (2018, March 04). Retrieved March 06, 2018, from https://en.wikipedia.org/wiki/Liquid_crystal
- Saxton, R. (n.d.). Ralph Saxton. Retrieved March 24, 2018, from http://mathfac.math.uno.edu/~rsaxton/