MINIMAL SURFACE EQUATION

Audrey Karl

THE PDE

Formula, Terms, Unknowns

MINIMAL SURFACE

- equivalent to having zero mean curvature
- Shape who has the least amount of area needed to occupy space/ **Minimizes the amount of space needed to create enclosure**
- quasi-linear elliptic PDE
  - similar to Laplace's Equation (compared to the wave or heat equation), but its analysis is much harder

MINIMIZES THE AREA FOR A GIVEN SURFACE CONSTRAINT

- For a smooth surface in R^{n+1} representing $x_{n+1} = u(x_1, \ldots, x_n)$ defined on a bounded open set Ω in R^n

-If u minimizes A(u) in U_g then equation 1 becomes the minimal surface equation

Source https://people.maths.ox.ac.uk/trefethen/pdectb/minsurf2.pdf

$\mathbb{R}^n$. Assuming that u is sufficiently smooth, the area of the surface is given by the nonlinear functional

$$\mathcal{A}(u) = \int_\Omega \left(1 + |\nabla u|^2\right)^{1/2} \, \mathrm{d}x_1 \ldots \mathrm{d}x_n, \qquad (1)$$

where ∇u is the gradient vector $(\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ and $|\nabla u|^2 = (\nabla u) \cdot (\nabla u)$.

The *minimal surface problem* is the problem of minimising $\mathcal{A}(u)$ subject to a prescribed boundary condition $u = g$ on the boundary $\partial\Omega$ of Ω. To do this, we consider the set $\mathcal{U}_g$ of all (sufficiently smooth) functions defined on $\bar{\Omega}$ that are equal to g on $\partial\Omega$. A classical result from the calculus of variations asserts that if u is a minimiser of $\mathcal{A}(u)$ in $\mathcal{U}_g$, then it satisfies the Euler–Lagrange equation

$$\nabla \cdot \left(\nabla u \big/ (1 + |\nabla u|^2)^{1/2}\right) = 0. \qquad (2)$$

This quasi-linear elliptic PDE is known as the *minimal surface equation*.

MANY DEFINITIONS

- Minimal surfaces can be defined in several equivalent ways in $\mathbf{R}^3$
- All definitions are equivalent
- They show how this equation applies to many different fields of math, including:
  - differential geometry
  - calculus of variations
  - potential theory
  - complex analysis
  - mathematical physics

DIFFERENTIAL EQUATION DEFINITION

- A surface $M \subset \mathbf{R}^3$ is minimal if and only if it can be locally expressed as the graph of a solution of
  - $(1+ u_x^2)\, u_{yy} - 2\, u_x\, u_y\, u_{xy} + (1+ u_y^2)\, u_{xx} = 0$

- Originally found in 1762 by Lagrange
- In 1776, Jean Baptiste Meusnier discovered that it implied a **vanishing mean curvature**

MANY DEFINITIONS

Local Least Area

A surface $M \subset \mathbf{R}^3$ is minimal if and only if **every point p ∈ M has a neighborhood with least-area relative to its boundary.**

This property is local: there might exist other surfaces that minimize area better with the same global boundary

Variational

A surface $M \subset \mathbf{R}^3$ is minimal if and only **if it is a critical point of the area functional for all compactly supported variations.**

This makes minimal surfaces a **2-dimensional analogue to geodesics**

Mean Curvature

A surface $M \subset \mathbf{R}^3$ is minimal if and only if **its mean curvature vanishes identically.**

Direct implication: **every point on the surface is a saddle point** with equal and opposite principal curvatures.

SOAP FILM DEFINITION

- A surface $M \subset \mathbf{R}^3$ is minimal if and only if **every point $p \in M$ has a neighborhood D_p** which is equal to the unique idealized soap film with boundary ∂D_p
- By the Young–Laplace equation the **curvature of a soap film is proportional to the difference in pressure between the sides**: if it is zero, the membrane has zero mean curvature.
  - Note that **spherical bubbles are *not* minimal surfaces as per this definition**: while they minimize total area subject to a constraint on internal volume, they have a positive pressure.

MANY MORE DEFINITIONS

- Energy Definition
  - Ties minimal surfaces to harmonic functions and potential theory
- Harmonic Definition
  - Implies that the maximum principle for harmonic functions is that there are no compact complete minimal surfaces in $\mathbf{R}^3$.
- Gauss Map Definition
  - links the mean curvature to the derivatives of the Gauss map and Cauchy–Riemann equations.
- Mean Curvature Flow Definition - Minimal surfaces are the critical points for the mean curvature flow
- Local Least Area and Variational Definitions
  - allow extending minimal surfaces to other Riemannian manifolds than $\mathbf{R}^3$.

HISTORY

JOSEPH ANTOINE FERDINAND PLATEAU (1801-1883)

- The minimal surface problem is also known as the **classical Plateau problem**
- He experimented by dipping wire contours into solutions of soapy water and glycerin
  - Although he did not have the mathematical skills to investigate quantitatively, he theorized much with bubble blowing

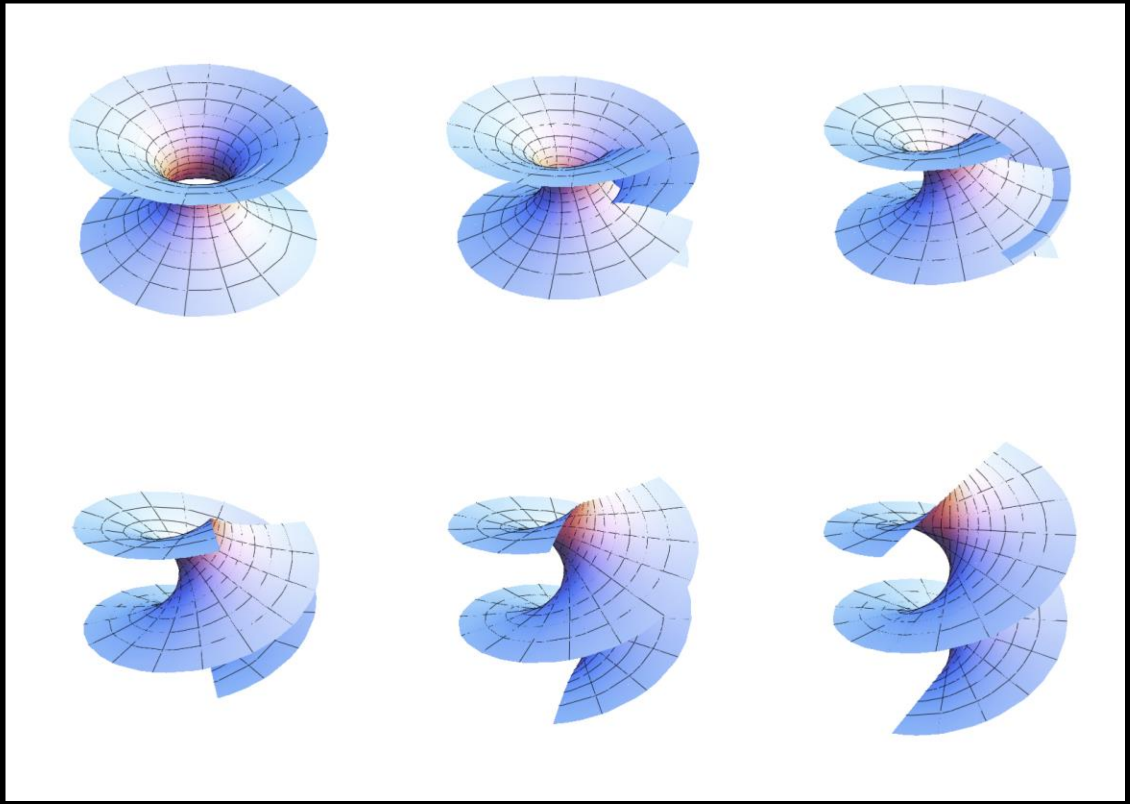
LAGRANGE 1762

- Attempted to find the surface $z = z(x, y)$ of **least area stretched across a given closed contour**
  - Derived the Euler–Lagrange equation for the solution
  - Did not find a solution beyond the plane (the trivial solution)

$$\frac{d}{dx}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{d}{dy}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=0$$

JEAN BAPTISTE MARIE MEUSNIER 1776

- Discovered that the helicoid and catenoid satisfy Lagrange’s equation https://www.youtube.com/watch?v=E6JtYMVayeI
- The differential expression corresponds to twice the mean curvature of the surface
    - Conclusion: **surfaces with zero mean curvature are area-minimizing**.



GASPARD MONGE AND LEGENDRE 1795

- Derived representation formulas for the solution surfaces
  - Successfully used by Heinrich Scherk in 1830 to derive his surfaces
    - These were generally regarded as practically unusable at the time

Others:

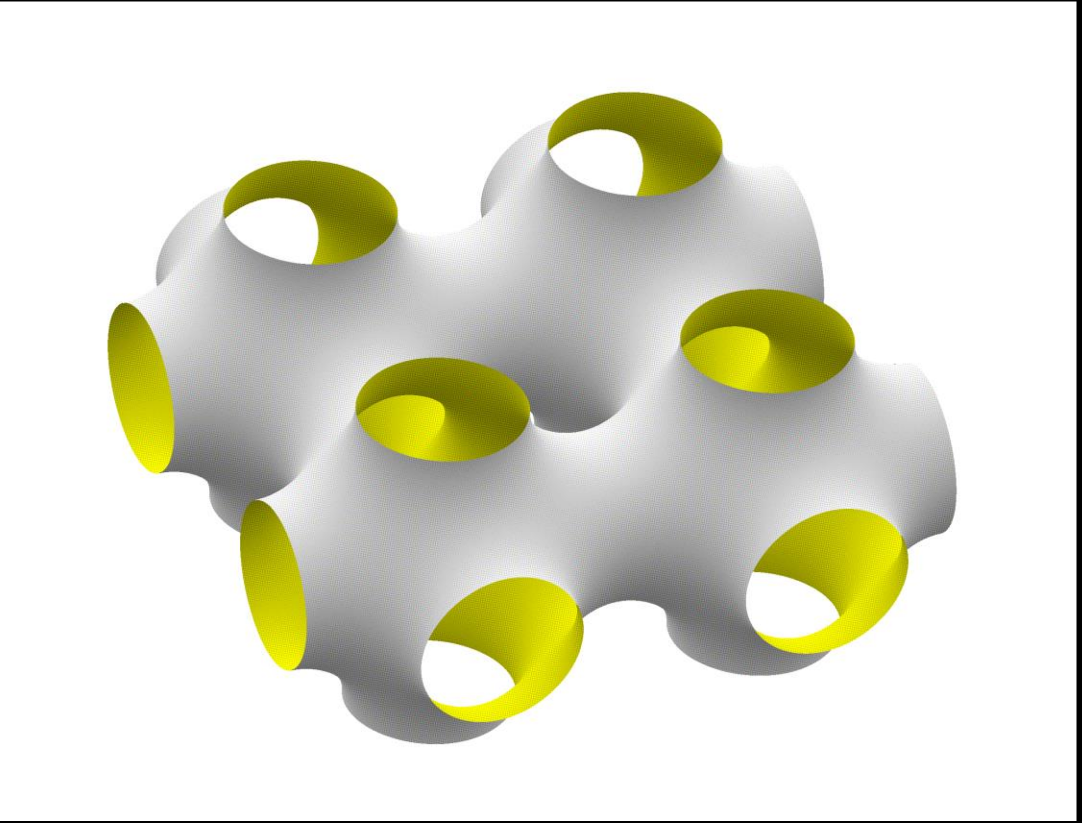
- **Catalan** proved in 1842/43 that the helicoid is the only ruled minimal surface.
- Later there were many other important contributions from **Schwarz, Beltrami, Bonnet, Darboux, Lie, Riemmann, Serret, and Weingarten**

CELSO COSTA 1982

- Disproved the idea that the **plane, the catenoid, and the helicoid** were the only complete embedded minimal surfaces in $\mathbf{R}^3$ of finite topological type.
  - **Stimulated new work** on using the old parametric methods
  - Demonstrated the **importance of computer graphics** to visualize the studied surfaces and numerical methods to solve the "period problem"

HERMANN KARCHER 1989

- Proved existence of **triply periodic minimal surfaces**
  - originally described empirically by Alan Schoen in 1970
- Led to many **new surface families and methods of deriving new surfaces from old**
  - for example by adding handles or distorting them
- The Schwarz P surface →

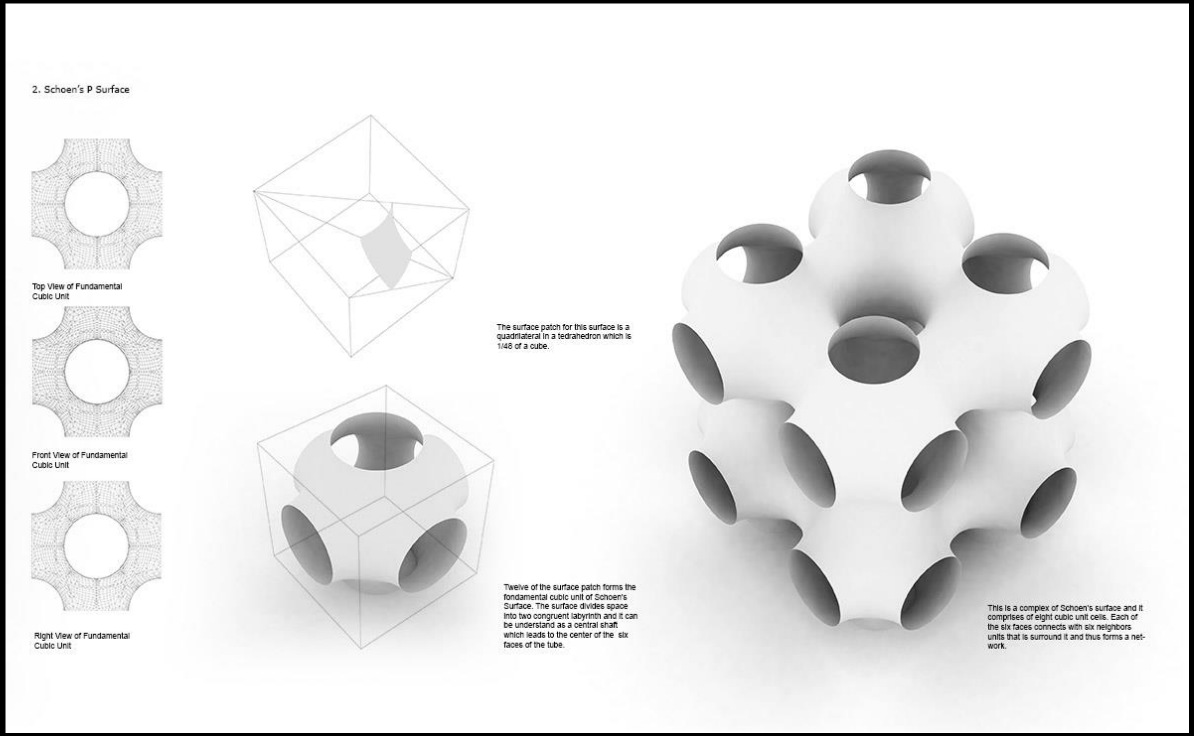


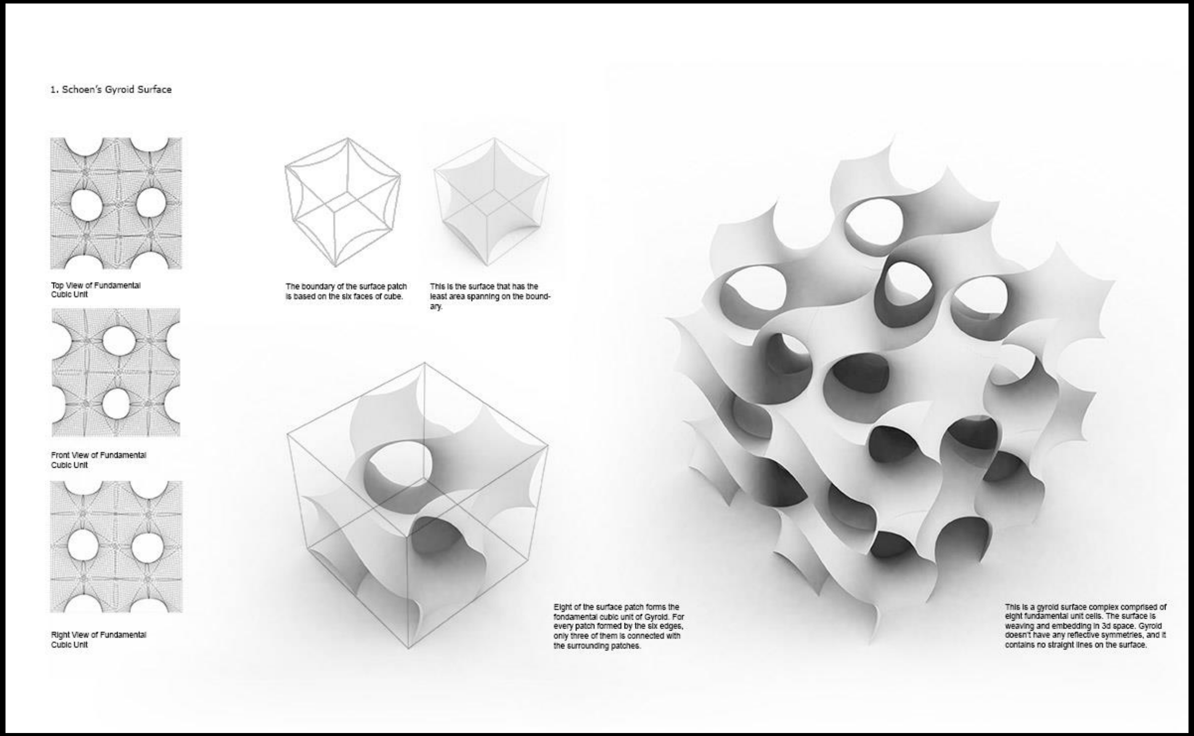
SOME MODERN SURFACES INCLUDE

- **The Gyroid**: One of Schoen's 1970 surfaces, a triply periodic surface of particular interest for liquid crystal structure
- **The Saddle Tower Family**: generalizations of Scherk's second surface
- **Costa's Minimal Surface**
  - Famous conjecture **disproof** of the idea that the plane, helicoid and the catenoid were the only embedded minimal surfaces that could be formed by puncturing a compact surface
  - Jim Hoffman, David Hoffman and William Meeks III then **extended the definition to produce a family of surfaces** with different rotational symmetries.
- **The Chen–Gackstatter Surface Family**
  - added handles to the Enneper surface.

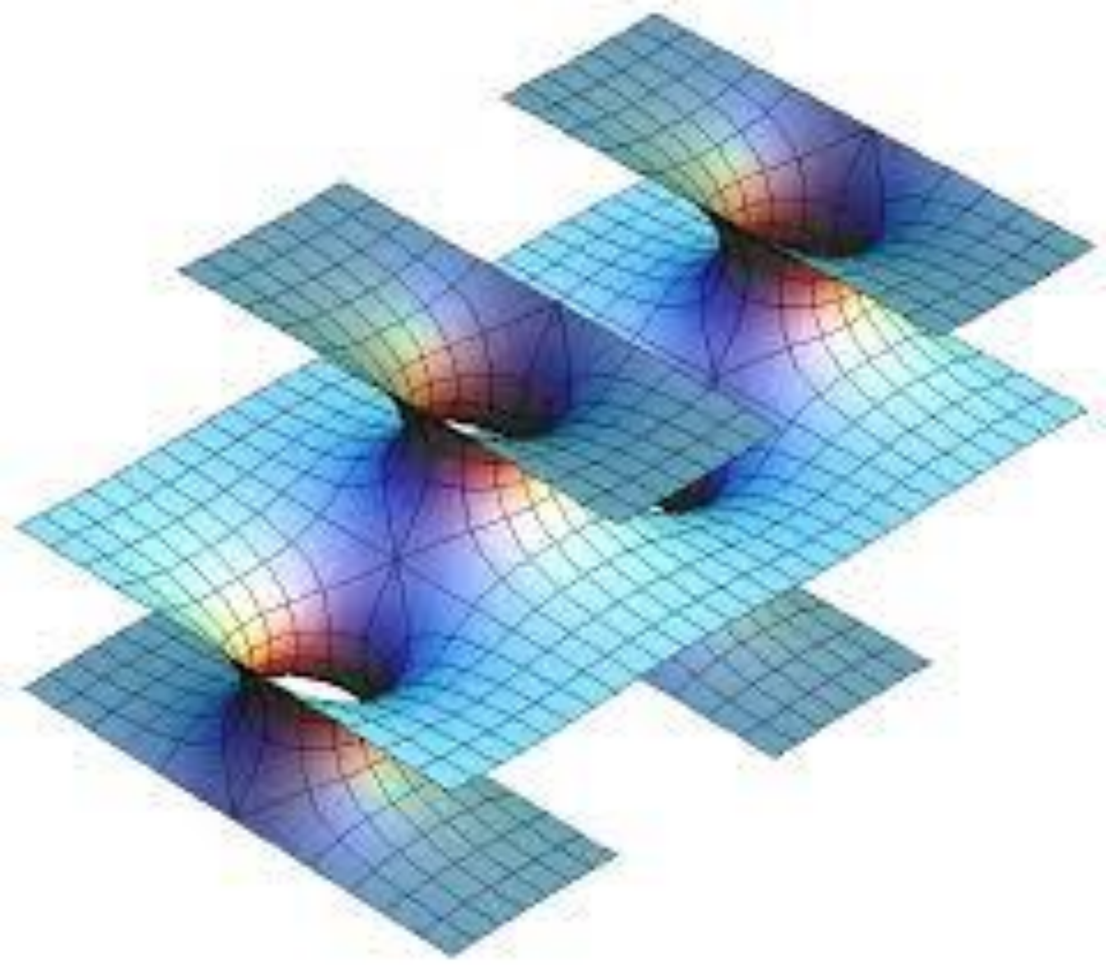
SOME EXAMPLES OF GYROIDS

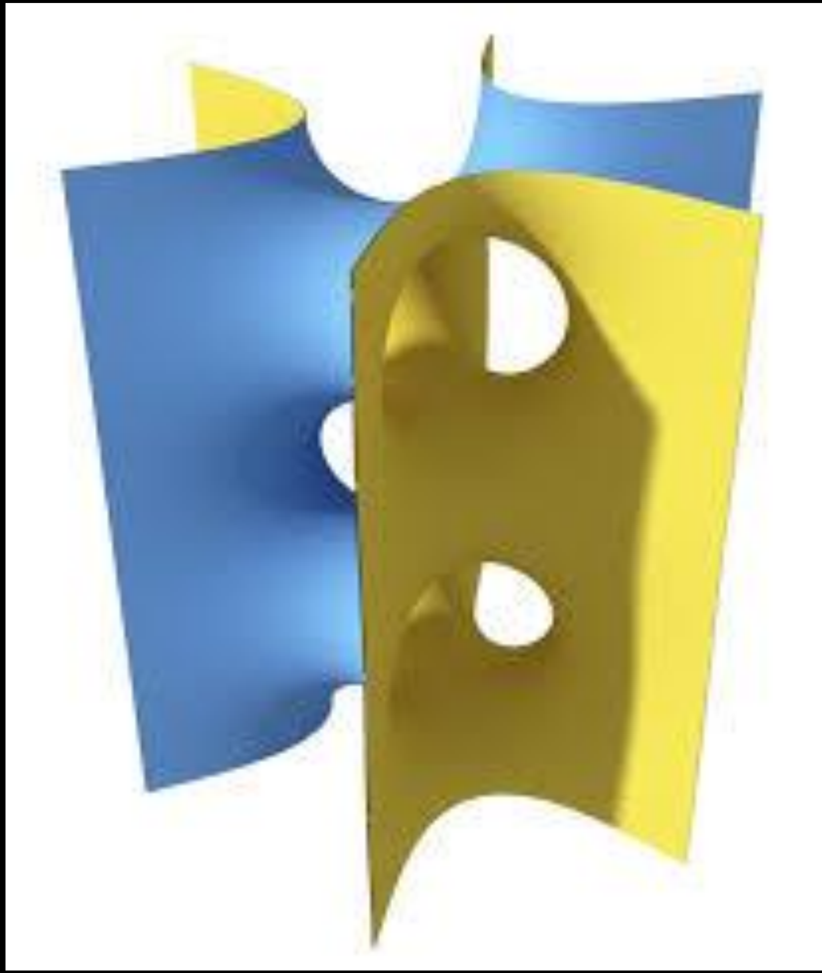
(gyro meaning “a circle” and “-id” for "belonging to")



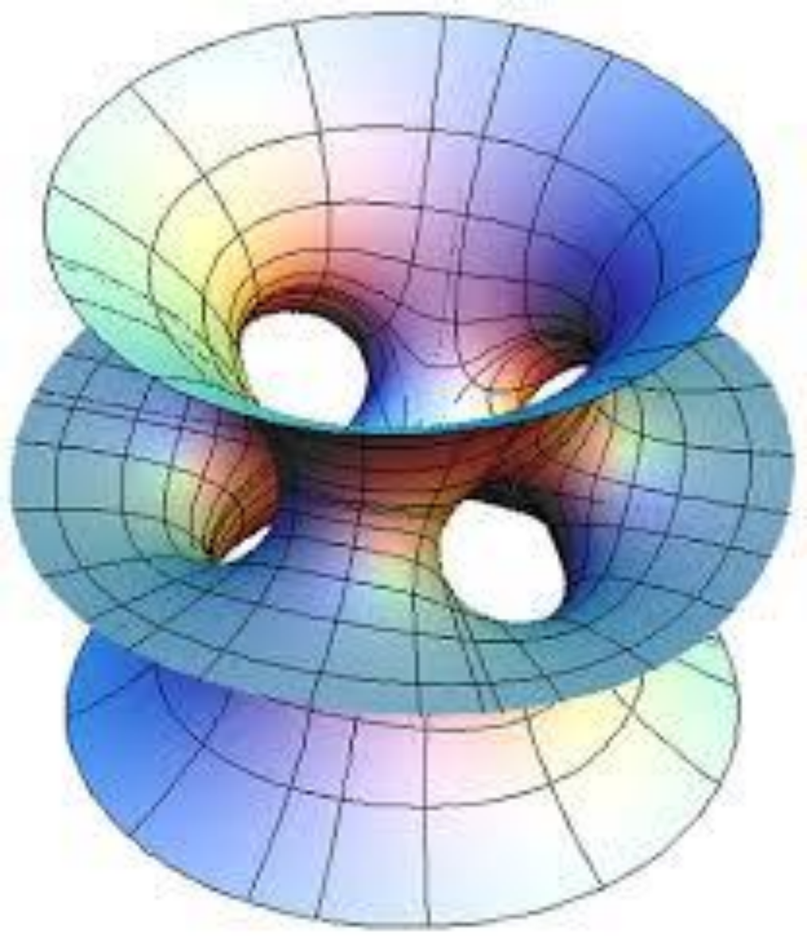


THE SADDLE TOWER



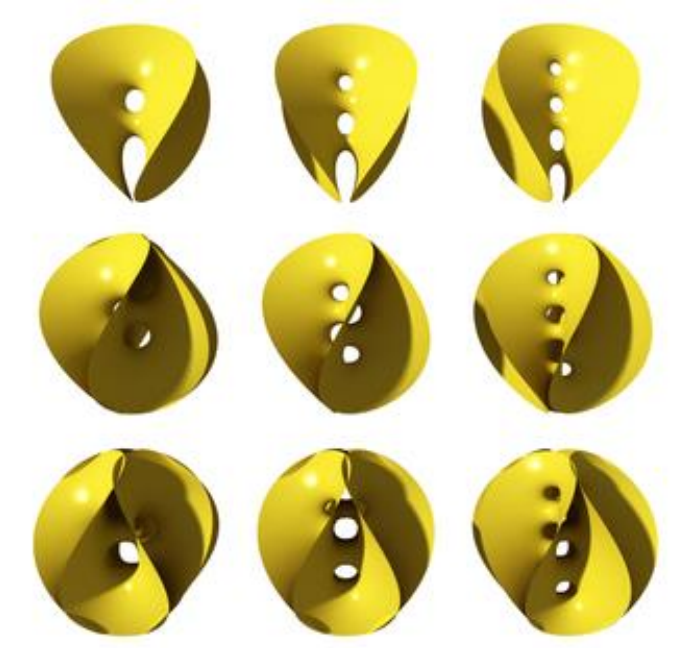


COSTA'S MINIMAL SURFACE

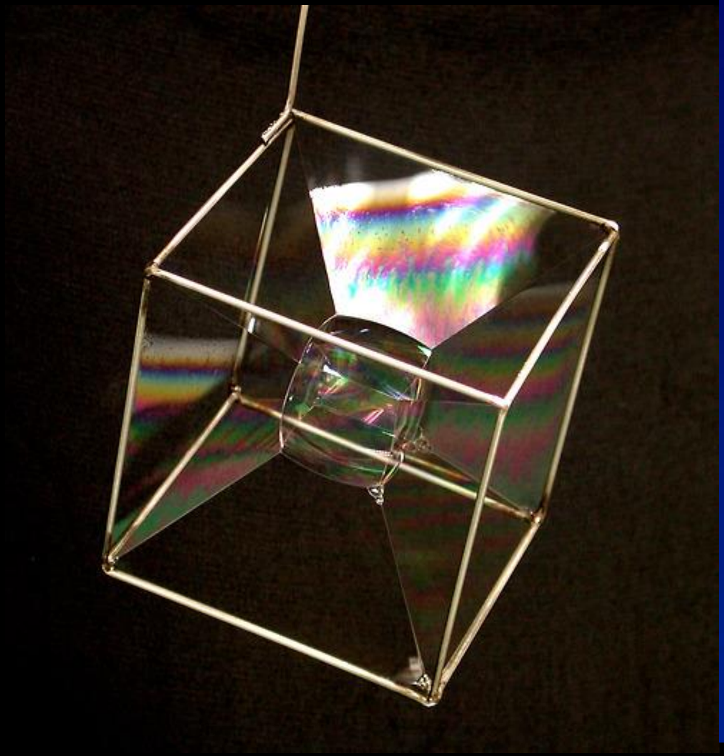


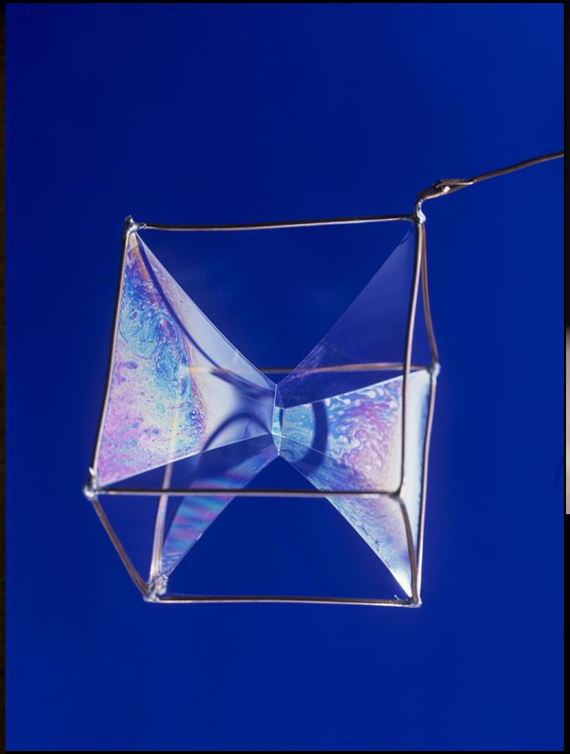
CHEN-GACKSTATTER SURFACE

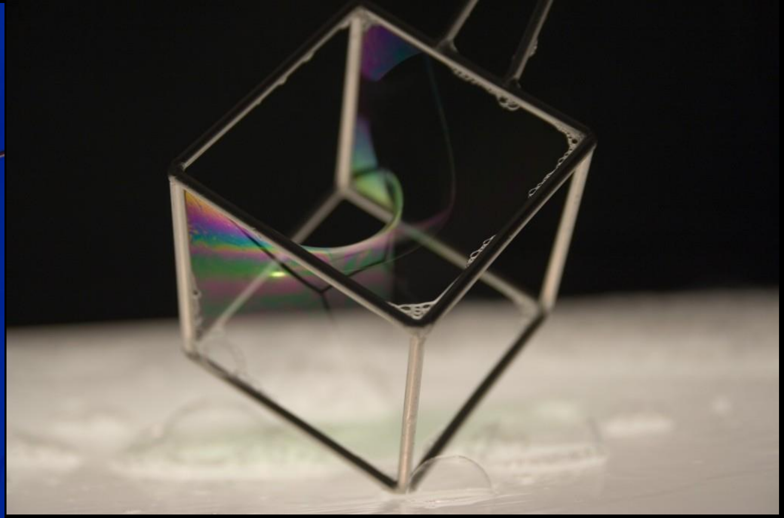
The First 9 CG Surfaces



APPLICATIONS





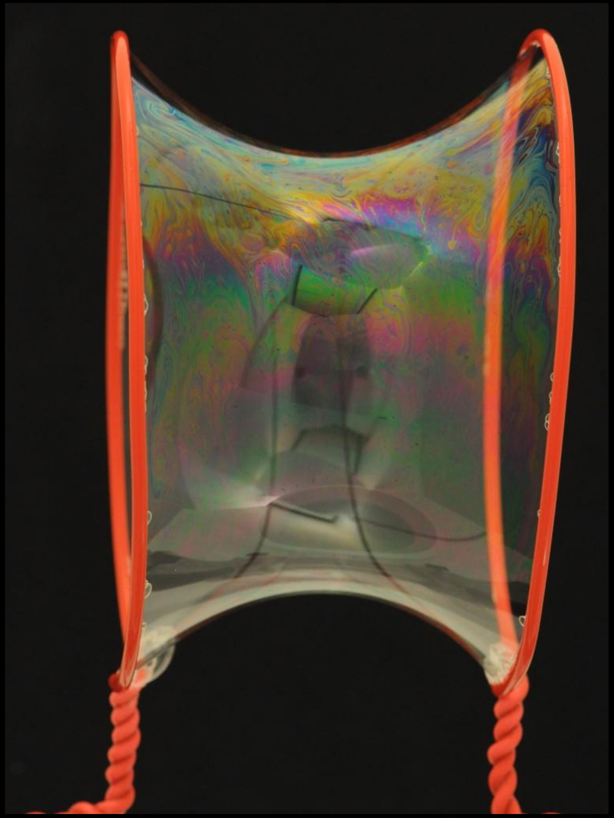


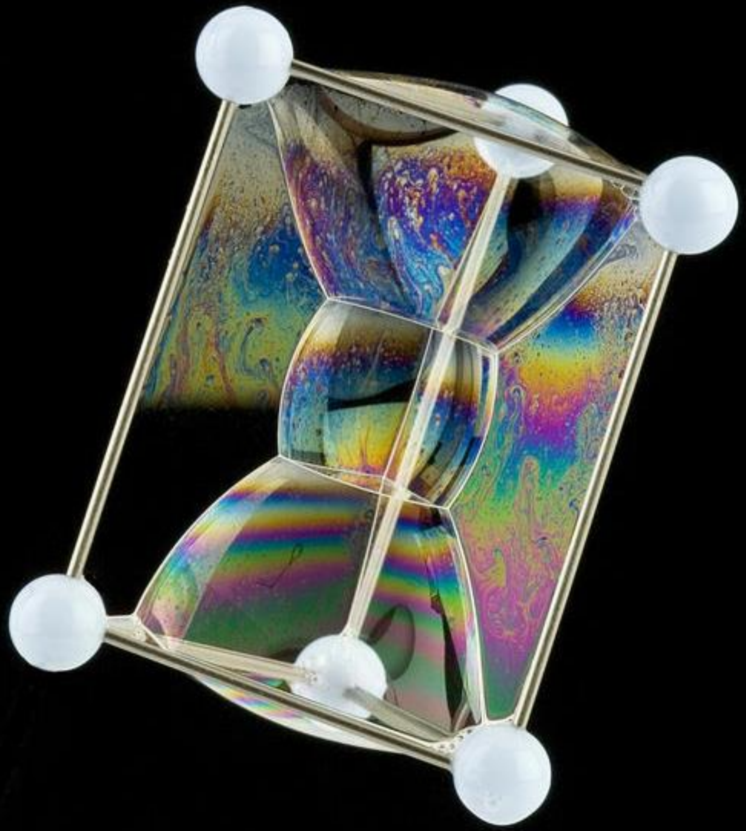
BUBBLES!

The minimal surface problem is also known as the classical Plateau problem, after the Belgian physicist Joseph Antoine Ferdinand Plateau (1801-1883)

NEW SHAPES AND PATTERNS

New shapes and patterns to build off of

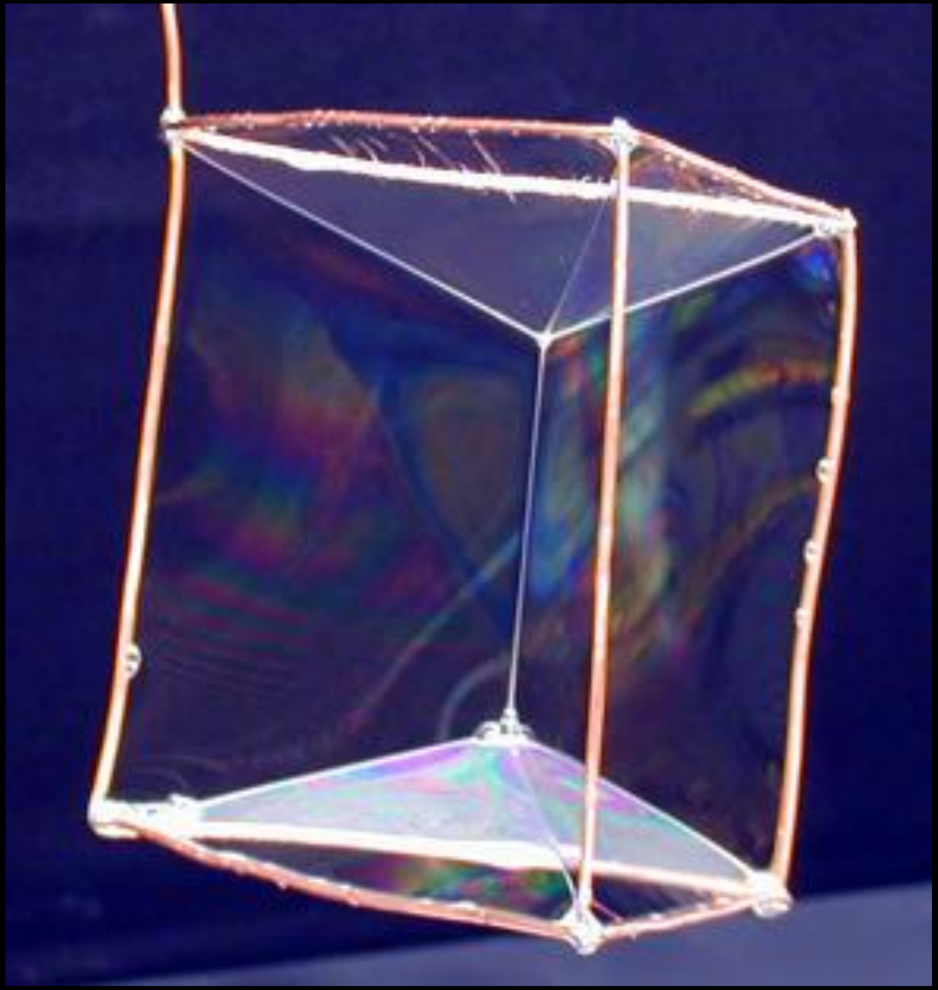




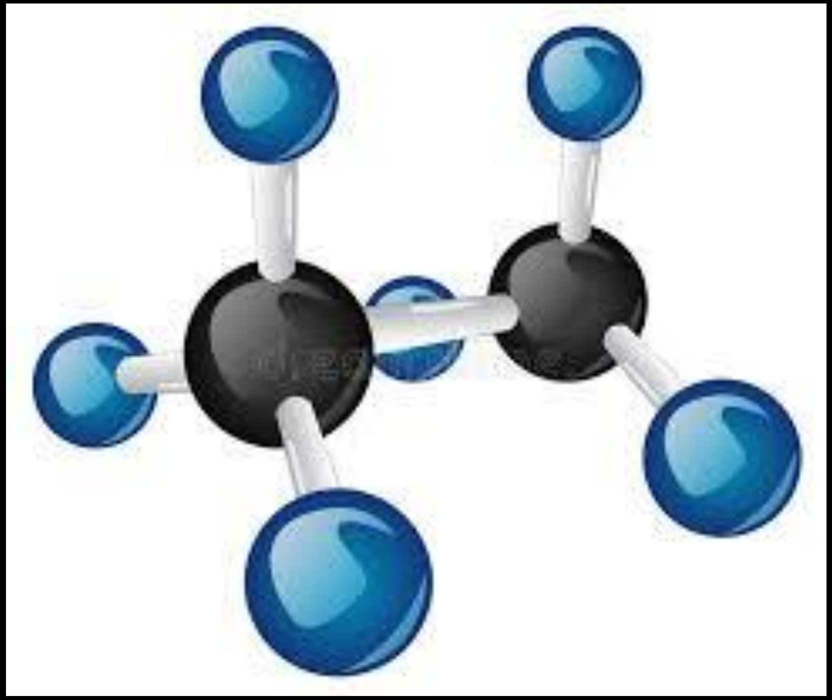
MANY FIELDS OF ENGINEERING AND SCIENCE

- **Discrete differential geometry** discrete minimal surfaces are studied
  - often used in this field to **approximate minimal surfaces numerically**, even if no closed form expressions are known.
- **Brownian motion studies** on minimal surfaces lead to **probabilistic proofs** of several theorems on minimal surfaces.
- **Molecular Engineering and Materials Science**
  - anticipated applications in **self-assembly of complex materials**
- **General relativity**
  - Links to the **theory of black holes** and the Plateau problem
- **Art and Design**

MODELING MOLECULAR INTERACTIONS

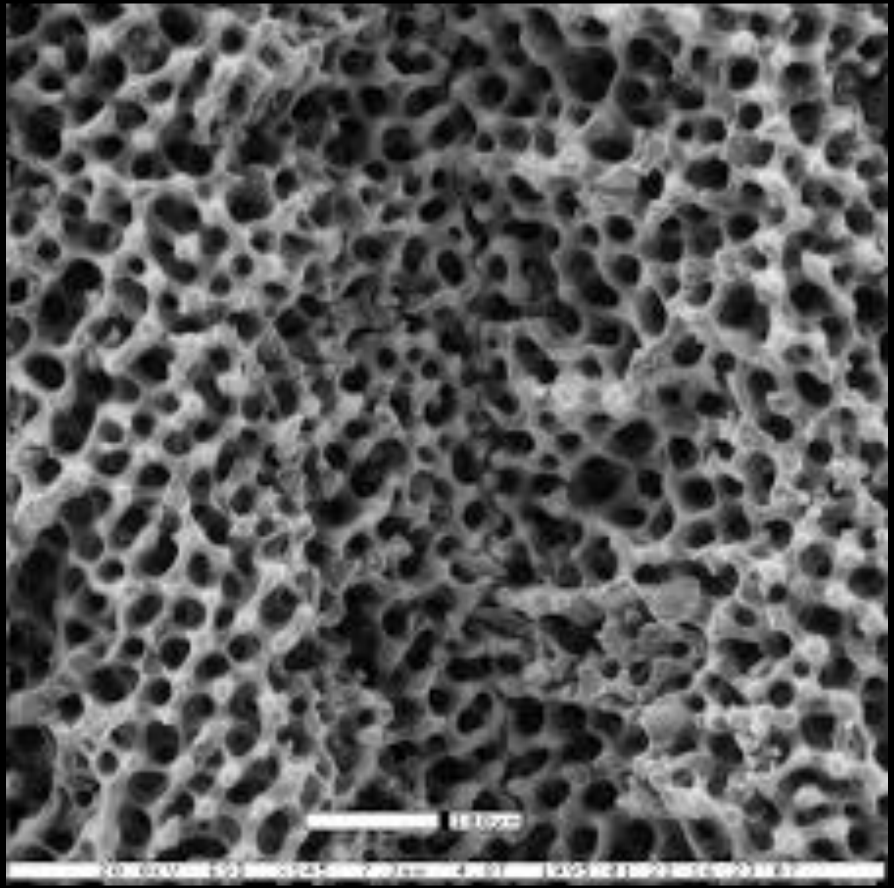


Angles that form are identical to bond angles



GROWING CRYSTALS

- Calcite particles can be "grown" in surface treated polymer membranes that can be seen under an electron microscope
- Demonstrates a natural yet primitive minimal surface shape (a gyroid) for its efficiency in compression rather tension like we're used to seeing at visible scales.



ART AND ARCHITECTURE

- Costa’s minimal surface

ARCHETECTURE

The Olympiapark in Münich by Frei Otto was inspired by soap surfaces.

FUTURE OF THE FIELD

Theoretical and Digital

Many applications to

- Lie groups and Lie algebras
- Homologies and cohomologies
- Bordisms and varifolds
- Computer visualization and animation techniques

Physical

- Have relevance to a variety of problems in materials science and civil engineering.