Navier-Stokes Equations

Some background:

Claude-Louis Navier was a French engineer and physicist who specialized in mechanics

Navier formulated the general theory of elasticity in a mathematically usable form (1821), making it available to the field of construction with sufficient accuracy for the first time. In 1819 he succeeded in determining the zero line of mechanical stress, finally correcting Galileo Galilei's incorrect results, and in 1826 he established the elastic modulus as a property of materials independent of the second moment of area. Navier is therefore often considered to be the founder of modern structural analysis.

His major contribution however remains the Navier–Stokes equations (1822), central to fluid mechanics.

His name is one of the 72 names inscribed on the Eiffel Tower

Sir George Gabriel Stokes was an Irish physicist and mathematician. Born in Ireland, Stokes spent all of his career at the University of Cambridge, where he served as Lucasian Professor of Mathematics from 1849 until his death in 1903. In physics, Stokes made seminal contributions to fluid dynamics (including the Navier–Stokes equations) and to physical optics. In mathematics he formulated the first version of what is now known as Stokes's theorem and contributed to the theory of asymptotic expansions. He served as secretary, then president, of the Royal Society of London

The stokes, a unit of kinematic viscosity, is named after him.

Navier Stokes Equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u},$$

the simplest form, where $\mathbf{u}$ is the fluid velocity vector, P is the fluid pressure, and ρ is the fluid density, ν is the kinematic viscosity, and ∇^2 is the Laplacian operator.

In physics, the Navier–Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, each of whom derived these equations independently, are a set of nonlinear partial differential equations which describe the motion of viscous fluids and are the fundamental equations of fluid dynamics. These equations result from applying Newton's second law to fluid dynamics, along with the assumption that the stress in the fluid is the sum of a diffusing viscous term (based on the way that the velocity is changing) and a pressure term, describing viscous flow.

The Navier–Stokes equations are based on the work of Leonhard Euler (1707–1783). Euler considered the fluid as a continuum allowing him to derive governing equations for the motion of inviscid (non-viscous) fluids based on differential calculus. His equations were the first written nonlinear partial differential equations—the Euler equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla P}{\rho}$$

, where u is the fluid velocity vector, P is the fluid pressure, ρ is the fluid density, and ∇ indicates the gradient differential operator.

In 1821 French engineer Claude-Louis Navier introduced the element of viscosity (friction) for the more realistic and vastly more difficult problem of viscous fluids. Throughout the middle of the 19th century, British physicist and mathematician Sir George Gabriel Stokes improved on this work, though complete solutions were obtained only for the case of simple two-dimensional flows. The complex vortices and turbulence, or chaos, that occur in three-dimensional fluid (including gas) flows as velocities increase have proven intractable to any but approximate numerical analysis methods.

Stokes and Navier each contributed a viscous diffusion term to account for the viscosity of a fluid.

The Navier Stokes Equations can be written and expressed in several different ways to account for different environmental factors, dimensions, and varying levels of complexity involved in any given situation.

Fluid dynamics deals with the motion of liquids and gases, which when studied macroscopically, appear to be continuous in structure. All the variables are considered to be continuous functions of the spatial coordinates and time. For irrotational flow, expressed as

$$\nabla \times \mathbf{u} = \left(\frac{\partial \mathbf{u}_z}{\partial y} - \frac{\partial \mathbf{u}_y}{\partial z}\right) i + \left(\frac{\partial \mathbf{u}_x}{\partial z} - \frac{\partial \mathbf{u}_z}{\partial x}\right) j + \left(\frac{\partial \mathbf{u}_y}{\partial x} - \frac{\partial \mathbf{u}_x}{\partial y}\right) k = 0,$$

the Navier-Stokes equations, in 2D can be written as:

The continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0 \qquad (13)$$

The *U*-momentum equation:

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} + \frac{\partial(\rho VU)}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right] + \rho F_x \qquad (14)$$

The *V*-momentum equation:

$$\frac{\partial(\rho V)}{\partial t} + \frac{\partial(\rho UV)}{\partial x} + \frac{\partial(\rho V^2)}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial V}{\partial y}\right] + \rho F_y$$

In 3D, things are a little more complicated:

The Navier-Stokes equations consists of a time-dependent continuity equation for conservation of mass, three time-dependent conservation of momentum equations and a time-dependent conservation of energy equation. There are four independent variables in the problem, the x, y, and z spatial coordinates of some domain, and the time t. There are six dependent variables; the pressure p, density r, and temperature T (which is contained in the energy equation through the total energy Et) and three components of the velocity vector; the u component is in the x direction, the v component is in the y direction, and the w component is in the z direction, All of the dependent variables are functions of all four independent variables.. Together with the equation of state such as the ideal gas law - p V = n R T, the six equations are just enough to determine the six dependent variables. In general, all of the dependent variables are functions of all four independent variables. Usually, the Navier-Stokes equations are too complicated to be solved in a closed form. However, in some special cases the equations can be simplified and may admit analytical solutions.

Navier-Stokes Equations

3 - dimensional - unsteady

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Coordinates: (x,y,z)
Velocity Components: (u,v,w)

Time : t Pressure: p
Density: ρ Stress: τ
Total Energy: Et

Heat Flux: q
Reynolds Number: Re
Prandtl Number: Pr

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

X – Momentum:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re_r}\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right]$$

Y – Momentum:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{Re_r}\left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right]$$

Z – Momentum

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re_r}\left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

Energy:

$$\frac{\partial(E_T)}{\partial t} + \frac{\partial(uE_T)}{\partial x} + \frac{\partial(vE_T)}{\partial y} + \frac{\partial(wE_T)}{\partial z} = -\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} - \frac{1}{Re_r Pr_r}\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right]$$
$$+ \frac{1}{Re_r}\left[\frac{\partial}{\partial x}(u\,\tau_{xx} + v\,\tau_{xy} + w\,\tau_{xz}) + \frac{\partial}{\partial y}(u\,\tau_{xy} + v\,\tau_{yy} + w\,\tau_{yz}) + \frac{\partial}{\partial z}(u\,\tau_{xz} + v\,\tau_{yz} + w\,\tau_{zz})\right]$$

, where Re is the Reynolds number which is a similarity parameter that is the ratio of the scaling of the inertia of the flow to the viscous forces in the flow. The q variables are the heat flux components and Pr is the Prandtl number which is a similarity parameter that is the ratio of the viscous stresses to the thermal stresses. The tau variables are components of the stress tensor. A tensor is generated when you multiply two vectors in a certain way. Our velocity vector has three components; the stress tensor has nine components. Each component of the stress tensor is itself a second derivative of the velocity components.

The terms on the left hand side of the momentum equations are called the convection terms of the equations. Convection is a physical process that occurs in a flow of gas in which some property is transported by the ordered motion of the flow. The terms on the right hand side of the momentum equations that are multiplied by the inverse Reynolds number are called the diffusion terms. Diffusion is a physical process that occurs in a flow of gas in which some property is transported by the random motion of the molecules of the gas. Diffusion is related to the stress tensor and to the viscosity of the gas. Turbulence, and the generation of boundary layers, are the result of diffusion in the flow. The Euler equations contain only the convection terms of the Navier-Stokes equations and can not, therefore, model boundary layers. There is a special simplification of the Navier-Stokes equations that describe boundary layer flows.

Notice that all of the dependent variables appear in each equation. To solve a flow problem, you have to solve all five equations simultaneously; that is why we call this a coupled system of equations.

The Navier–Stokes equations are very useful because they describe the physics of many different scientific phenomena and are widely used in both science and engineering. Scientists and engineers use the equations in mathematical models of weather, ocean currents, water flow in a pipe, air flow around a wing, drag in race cars, optimizing particle filters, studying environmental particle transport, how stars move inside a galaxy, and much more. The Navier–Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Together with Maxwell's equations (the equations for electricity and magnetism) they can be used to model and study how things that can flow and conduct electricity can produce (and react to) magnetic fields.

The Navier–Stokes equations dictate not position but rather velocity (how fast the fluid is going and where it is going). A solution of the Navier–Stokes equations is called a velocity field or flow field, which is a description of the velocity of the fluid at a given point in space and time. It is a field, since it is defined at every point in a region of space and an interval of time. Once the velocity field is solved for, other quantities of interest (such as pressure. temperature, flow rate or drag force) may be found. This is different from what one normally sees in classical mechanics, where solutions are typically trajectories of position of a particle or deflection of a continuum. Studying velocity instead of position makes more sense for a fluid, but for visualization purposes one also can compute various paths that a particle could flow along.

The Navier–Stokes equations are also of great interest in a purely mathematical sense. Surprisingly, despite their wide range of practical uses it has not yet been proven that in three dimensions solutions always exist, or that if they do exist, then they are smooth, i.e. they do not contain any mathematical singularity. These are called the Navier–Stokes existence and smoothness problems. The Clay Mathematics Institute of Cambridge, MA has called this one of the seven most important open problems in mathematics, called a Millenium problem, and has offered a US$1,000,000 prize for a solution or a counterexample.

In 2008 the U.S. Defense Advanced Research Projects Agency (DARPA) listed it as one of the DARPA Mathematical Challenges, 23 mathematical problems for which it was soliciting research proposals for funding—“Mathematical Challenge Four: 21st Century Fluids. Classical fluid dynamics and the Navier-Stokes Equation were extraordinarily successful in obtaining quantitative understanding of shock waves, turbulence, and solitons, but new methods are needed to tackle complex fluids such as foams, suspensions, gels, and liquid crystals.

Optional: Story by Jeremy Teitelbaum, Dean of CLAS

Friday morning, exactly as predicted, the first flakes of snow began to fall near my house in Coventry, Conn. The millions of us who live in the Northeast Corridor were prepared for a potentially record-breaking blizzard, and over the next 24 hours we watched that blizzard do exactly what the forecasters had predicted.

Accounting for Hurricane Sandy, this was the second time in the past six months that forecasters predicted catastrophic events with astonishing accuracy, saving lives and preventing even more devastating property damage. I think it appropriate to celebrate their achievements. I therefore propose creating a new holiday, to be celebrated every year on Feb. 8, the anniversary of this storm, and to be called National Numerical Solutions to the Navier-Stokes Partial Differential Equations Day.

The Navier-Stokes partial differential equations, named after their discoverers, the 19th-century mathematicians George Gabriel Stokes and Claude-Louis Navier, govern the motion of a general fluid. Although one can write the equations very compactly using modern mathematical notation, they contain within them the full range of turbulent behavior that occurs in moving fluids, in settings as varied as airflow over a wing, water waves on a beach, and, of course, the weather.

The complexity of the Navier-Stokes equations means that one cannot hope to write down solutions for them. Instead, scientists and engineers use computer programs to construct approximations to solutions; the “European Model” that has been cited recently in weather forecasts is such a computer program.

While there is no question that such a model is an interdisciplinary triumph, with physicists, engineers, computer scientists, statisticians, a whole range of environmental scientists, and mathematicians working together to collect data, write code, and test the output of the model against reality, without the Navier-Stokes equations there would be no place to begin.

One of the ironic features of the Navier-Stokes equations is that, despite their enormous influence in applied mathematics, they are not understood from a theoretical perspective. Confronted with a system of partial differential equations, the first question that mathematicians ask about them is whether or not one can be sure that the equations have a solution in the first place. In the case of the Navier-Stokes equations, the question of whether or not solutions always exist is unsolved. Indeed, establishing the existence (or non-existence) of such solutions is one of the seven Clay Foundation Millennium problems, and solving it carries a prize of $1,000,000.

Whether or not we can solve the mathematical problem of the existence of solutions to Navier-Stokes, we can see the existence of solutions all around us in the working of the weather and the flow of water. The astonishing accuracy of weather forecasting that we are witnessing is another vindication of what Eugene Wigner called “the unreasonable effectiveness of mathematics in the physical sciences.”

Given the life-saving power of those forecasts, as well as the many other technological advances that stem from the ever-improving ability to approximate solutions of the Navier-Stokes equations, I know that you’ll join me in calling for the establishment of National Numerical Solutions to the Navier-Stokes Partial Differential Equations Day.

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